

The University of Reading
ISMA Centre - Numerical Methods 2 Project Report
Quadratic Interest Rate Model Simulation

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1 Project Requirements

This project concerned the simulation of the quadratic interest rate model. The given project requirements were :

1. Simulate the short term rate $r(t)$.
2. Compute the corresponding bond price.
3. Compute the price of a European call option on the simulated bond price.
4. Compare the simulated call price with the closed form formulas given by Jamshidian, 1996.

The simulation was to be done using Monte Carlo methods. No variance reduction techniques were required for this project. It was suggested that 1000 simulations, a maximum time horizon of 10 years and a time step of 1 month should be used in the simulations.

The following literature was issued along with the project brief :

Jamshidian, F., Bond, futures and option evaluation in the quadratic interest rate model, Applied Mathematical Finance, Vol. 3, 1996, pp. 93-115.

Glasserman, P., Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003, pp. 124, 132-133.

James, J., Webber, N., Interest Rate Modelling, John Wiley & Sons, 1st Edition, January 2000, Chapter 9.3.2 pp. 229-231.

Note that the relative efficiencies of the computer algorithms developed have not been considered in the project. Calibration of short rate model constants to match the term structure implied by observed zero coupon bond prices has not been included.

Please see attached CD with Excel simulation files :

Rebel_NM2P_CIR_MC_Simulation.xls

Rebel_NM2P_QUAD_MC_Simulation.xls

The CD also includes electronic copies of the issued literature and other related documents.

2 Short Rate Simulation Notation

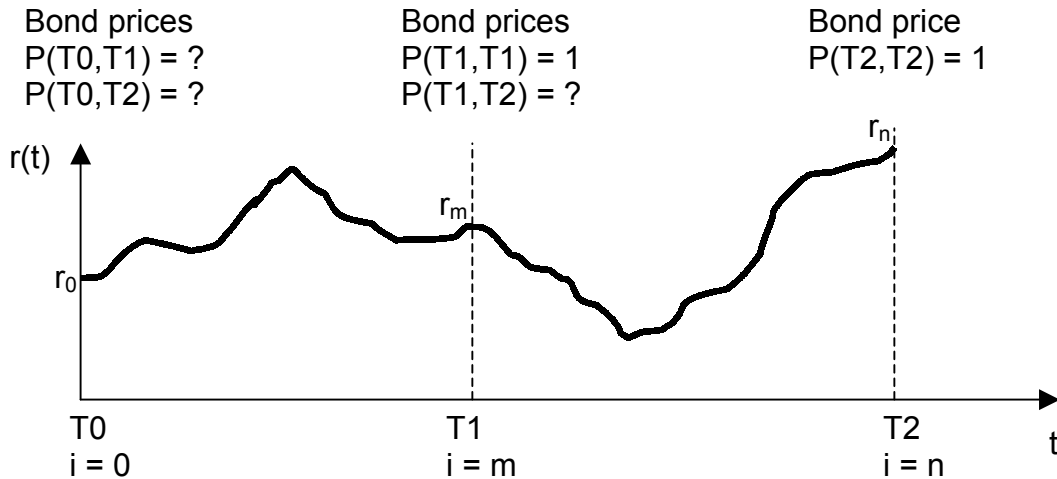


Figure 1 - General notation used in the short rate simulation, r_i , is the simulated short rate for each time point i .

Figure 1 shows the realisation of a short rate, $r(t)$, based on a single random path. Using the general short rate notation shown in Figure 1, the price of a zero coupon bond with a face value of 1 which matures at time $T2$, viewed at time $T1$, is :

$$P(T1, T2) = \text{Exp}\left(-\int_{T1}^{T2} r(t) dt\right) \cong \text{Exp}\left(-\Delta t \sum_{i=m}^{n-1} r_i\right)$$

Similarly, the price of a zero coupon bond with a face value of 1 which matures at time $T1$, viewed at time $T0$, is :

$$P(T0, T1) = \text{Exp}\left(-\int_0^{T1} r(t) dt\right) \cong \text{Exp}\left(-\Delta t \sum_{i=0}^{m-1} r_i\right)$$

For each independent random path of $r(t)$ between $T0$ and $T2$ these two bond prices will also be random.

The price at time $T0$ of a European call option on the bond maturing at time $T2$, where the option expires at time $T1$, can be written as follows:

$$C(0) = E[P(T0, T1)] \cdot \text{MAX}[E[P(T1, T2)] - K, 0]$$

where K is the call option strike price. This is the discounted expected terminal value of the option where the discount factor is a bond price which is random and hence the expectation needs to be taken.

This notation is used in the Monte Carlo simulation implemented in the two Excel files :

Rebel_NM2P_CIR_MC_Simulation.xls

Rebel_NM2P_QUAD_MC_Simulation.xls

3 Monte Carlo Simulation Workbooks

3.1 Cox Ingersoll Ross Model

In order to establish a general methodology for the Monte Carlo simulation applied in this project, a simple Cox Ingersoll Ross model was first constructed as shown in the file :

Rebel_NM2P_CIR_MC_Simulation.xls

The results of the simulated bond prices $P(T_0, T_1)$, $P(T_0, T_2)$ and $P(T_1, T_2)$ were compared with the closed form solutions provided by Hull on pages 542 and 543 (Options, Futures, and Other Derivatives, Pearson Education, Fifth Edition, 2003).

Figure 2 below shows typical input and resulting output from the workbook calculations.

CIR Simulation Parameters		
r0	4.0%	initial short term interest rate
alpha	0.20	speed of mean reversion
b	5.0%	mean reversion interest rate level
sigma	10.0%	volatility of short term interest rate
n	36	integer time steps to bond maturity (gives T2 with dt)
m	24	integer time steps to option maturity (gives T1 with dt) note $m < n$
dt	1/12	size of the time step in years
K	0.75	strike price of bond for option valuation
num_sims	1000	number of Monte Carlo simulations

CIR Simulation Output		CIR Closed Form	
Call option price	0.1906	0.1906	European style call, price at time $T_0 = 0$
P_T0_T1	0.9198	0.9203	bond maturing at T_1 , viewed at T_0 (note that $P_{T1_T1} = 1$)
P_T0_T2	0.8808	0.8814	bond maturing at T_2 , viewed at T_0 (note that $P_{T2_T2} = 1$)
P_T1_T2	0.9572	0.9571	bond maturing at T_2 , viewed at T_1
rT1 simulated	4.341%		the expected value of $r(t)$ at $t = T_1$, i.e. at time step m
rT1 exact	4.330%		see Shreve: Stochastic Calculus and Finance, 6 October 1997, Section 15.7, Page 172
d	4.0000		should be an integer value, see Glasserman Page 133
2*alpha*b - sigma²	0.0100		should be greater than 0 for $r(t)$ to be positive at all times
T1	2.0000		years (option maturity)
T2	3.0000		years (bond maturity)

CIR simulation, see Glasserman Page 120, Equation 3.62

Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003

CIR short rate dynamics : $dr(t) = \alpha(b - r(t))dt + \sigma r(t)^{0.5}dW(t)$

CIR closed form bond price equations, see Hull Pages 542-543

Options, Futures, and Other Derivatives, Pearson Education, Fifth Edition, 2003

CTRL-ALT-F9 = recalculate whole workbook

Figure 2 - Sample input and output from the Rebel_NM2P_CIR_MC_Simulation.xls workbook.

3.2 Separable Quadratic Model (with Vasicek short rate dynamics)

The general Monte Carlo simulation approach was then applied to the case of a separable, multifactor, quadratic model as described by Jamshidian in the issued literature, Section 8, pages 104 to 106. The assumption was made that all coefficients in the Vasicek short rate dynamics equation were constant (i.e. not functions of time). This assumption allowed for the application of the closed form equations for zero coupon bond prices as specified by Jamshidian on page 106. The quadratic model simulation is contained in the workbook :

Rebel_NM2P_QUAD_MC_Simulation.xls

Figure 3 below shows typical input and output values for a 4 factor, separable quadratic model simulation.

QUAD Simulation Parameters		
x0	4.0%	initial short term Vasicek interest rate
alpha	5.0%	Vasicek mean reversion interest rate level
beta	0.20	speed of Vasicek mean reversion
sigma	10.0%	volatility of Vasicek short term interest rate
n	36	integer time steps to bond maturity (gives T2 with dt)
m	24	integer time steps to option maturity (gives T1 with dt) note m < n
dt	1/12	size of the time step in years (should read delta t)
K	0.75	strike price of bond for option valuation
num_sims	1000	number of Monte Carlo simulations
n_quad	4	number of Vasicek factors in the separable quadratic model

QUAD Simulation Output		QUAD Closed Form	
Call option price	0.1809	0.1801	European style call, price at time T0 = 0
P_T0_T1	0.9467	0.9455	bond maturing at T1, viewed at T0 (note that P_T1_T1 = 1)
P_T0_T2	0.8914	0.8903	bond maturing at T2, viewed at T0 (note that P_T2_T2 = 1)
P_T1_T2	0.9410	0.9404	bond maturing at T2, viewed at T1
rT1 simulated	5.19%		the expected value of r(t) at t = T1, i.e. at time step m
rT1 exact	5.14%		see Glasserman Pages 110 and 111, Stationary Version, $E[x(t)^2] = (E[x(t)])^2 + \text{VAR}[x(t)]$
T1	2.0000		years (option maturity, from m)
T2	3.0000		years (bond maturity from n)

Vasicek short rate simulation, see Glasserman Pages 108-110, Equation 3.39 and 3.46
Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003

Vasicek short rate dynamics : $dx_i(t) = (\alpha - \beta x_i(t))dt + \sigma_i dW(t)$

alpha, beta and sigma are all assumed to be known constants

Separable quadratic model : $r(t) = 0.5[x_1(t)^2 + x_2(t)^2 + \dots + x_{n_quad}(t)^2]$

Separable quadratic model (Vasicek) closed form bond price equations, see Jamshidian Pages 104-106
Bond, futures and option evaluation in the quadratic interest rate model, Applied Mathematical Finance, Vol. 3, 1996

CTRL-ALT-F9 = recalculate whole workbook

Figure 3 - Sample input and output from the Rebel_NM2P_QUAD_MC_Simulation.xls workbook.

Careful examination of the Jamshidian closed form bond price equations under the separable multifactor quadratic model on page 108 revealed that there is a mistake in the 1996 paper. The equation for h_{τ} has "... + gamma + beta)" which is incorrect, it should be "... + gamma - beta)". This makes a significant difference to the results as h_{τ} is used extensively in the other linked equations. With this correction made, the simulated Monte Carlo bond prices match the closed form Jamshidian prices as shown in Figure 3.

Jamshidian's closed form equation for h_{τ} on page 108 was compared with the James / Webber Equation 9.68 which is the same as Jamshidian's B_t1_T2 containing h_{τ} . This revealed the error in the sign of beta.

4 Description of Functions Developed

4.1 Cox Ingersoll Ross Model

The following functions are contained in the Main module of the file:

Rebel_NM2P_CIR_MC_Simulation.xls

Function Call_on_zero_bond_CIR(r0, K, alpha, b, sigma, n, m, dt, num_sims)

This is a function for pricing of a European call option on a zero coupon bond with a face value of 1 where the bond matures at time T2 and the call option matures at time T1, $T1 < T2$. In the discrete time framework, T1 is denoted by $m*dt$ and T2 by $n*dt$ where $m < n$. This implementation uses the CIR short rate dynamics as specified in Glasserman page 120, Equation 3.62 (Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003). The function returns the value of the call and the three zero coupon bond prices : P_{T0_T1} , P_{T0_T2} , P_{T1_T2} . It also returns the expected value of the short rate at time T1, $r(T1) = r_m$. The simulated $r(T1)$ can be compared to readily available closed form solutions as can P_{T0_T2} which is not needed for the option pricing but is still calculated for completeness. Note that the closed form value of $r(T1)$ is required for calculating the closed form value of the bond price P_{T1_T2} .

Inputs:

r0	= initial short term interest rate (decimal form %)
K	= strike price of the bond for option valuation (face value = 1)
alpha	= speed of mean reversion in the CIR model
b	= mean reversion interest rate level in the CIR model (decimal form %)
sigma	= volatility of short term interest rate (decimal form %)
n	= integer time steps to bond maturity (gives T2 with dt)
m	= integer time steps to option maturity (gives T1 with dt) note $m < n$
dt	= size of the time step (years)
num_sims	= number of Monte Carlo simulations

CIR short rate dynamics : $dr(t) = \alpha*(b - r(t))*dt + \sigma*r(t)^{0.5}*dW(t)$

Outputs:

Column vector with the following scalar values :

Call option price = $P_{T0_T1} * \text{MAX}(P_{T1_T2} - K, 0)$

P_{T0_T1} = price of bond maturing at T1, viewed at T0 (note that $P_{T1_T1} = 1$)

P_{T0_T2} = price of bond maturing at T2, viewed at T0 (note that $P_{T2_T2} = 1$)

P_{T1_T2} = price of bond maturing at T2, viewed at T1

Simulated value for $r(T1)$ $T1 = m*dt$

Function CIR_rtip1(alpha, b, rti, i, dt, sigma, d)

This is a function for implementation of Figure 3.5 of Glasserman, page 124 (Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003). Note that it is used in conjunction with the function Call_on_zero_bond_CIR(...) listed above. It simulates $r(t)$ under CIR short rate dynamics.

This function was coded for completeness but was not used in the final implementation of the CIR Monte Carlo simulation. It was replaced by the alternative discretisation method suggested by Glasserman on page 133.

Inputs:

rti	= simulated short term interest rate at time step i (decimal form %)
alpha	= speed of mean reversion in the CIR model
b	= mean reversion interest rate level in the CIR model (decimal form %)
sigma	= volatility of short term interest rate (decimal form %)
dt	= size of the time step (years)
i	= current time step
d	= $4 * b * \alpha / \sigma^2$

Output:

Scalar value

rti+1 = simulated short term interest rate at time step i+1 (decimal form %)

Function P_CIR(alpha, b, rt1, sigma, t1, T2)

This function is the implementation of the CIR closed form zero coupon bond price equations from Hull pages 542 to 543 (Options, Futures, and Other Derivatives, Fifth Edition, 2003).

Inputs:

alpha	= speed of mean reversion in the CIR model
b	= mean reversion interest rate level in the CIR model (decimal form %)
rt1	= closed form short term interest rate at time t1 (decimal form %) = $b + \text{EXP}(-\alpha * t1) * (r0 - b)$ Shreve: Stochastic Calculus and Finance, 6 Oct 1997, Section 15.7, Page 172
sigma	= volatility of short term interest rate (decimal form %)
t1	= time at which bond price is determined (years)
T2	= bond maturity (years)

The face value of the zero coupon bond is assumed to be 1.

CIR short rate dynamics : $dr(t) = \alpha * (b - r(t)) * dt + \sigma * r(t)^{0.5} * dW(t)$

Output:

Scalar value

P_t1_T2 = price of a zero coupon bond maturing at T2, viewed at t1

4.2 Separable Quadratic Model (with Vasicek short rate dynamics)

The following functions are contained in the Main module of the file:

Rebel_NM2P_QUAD_MC_Simulation.xls

Function Call_on_zero_bond_QUAD(x0, K, alpha, beta, sigma, n, m, dt, num_sims, n_quad)

The function is for pricing of a European call option on a zero coupon bond where the bond matures at time T2 and the call option matures at time T1, $T_1 < T_2$. In the discrete time framework, T1 is denoted by $m*dt$ and T2 by $n*dt$ where $m < n$. The short rate Monte Carlo simulation is for the separable quadratic model (with Vasicek short rate dynamics) as discussed by Jamshidian on pages 104-106 (Bond, futures and option evaluation in the quadratic interest rate model, Applied Mathematical Finance, Vol. 3, 1996).

This implementation uses the Vasicek short rate dynamics as modified from Glasserman (Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003) pages 108-111, Equations 3.39 and 3.46 to form factors in the quadratic model. The function returns the value of the call option and the three zero coupon bond prices : P_{T_0,T_1} , P_{T_0,T_2} , P_{T_1,T_2} . It also returns the expected value of the short rate at time T1, $r(T_1) = r_m$. The simulated $r(T_1)$ can be compared to readily available closed form solutions as can P_{T_0,T_2} which is not needed for the option pricing but is still calculated for completeness.

Inputs:

x0	= initial Vasicek short term interest rate (decimal form %)
K	= strike price of the bond for option valuation (face value = 1)
alpha	= Vasicek mean reversion interest rate level (decimal form %)
beta	= speed of mean reversion in the Vasicek model
sigma	= volatility of Vasicek short term interest rate (decimal form %)
n	= integer time steps to bond maturity (gives T2 with dt)
m	= integer time steps to option maturity (gives T1 with dt) note $m < n$
dt	= size of the time step (years)
num_sims	= number of Monte Carlo simulations
n_quad	= number of Vasicek factors in the separable quadratic model

Vasicek short rate dynamics : $dx_i(t) = (\alpha - \beta x_i(t)) * dt + \sigma * dW(t)$

Separable quadratic model : $r(t) = 0.5 * [x_1(t)^2 + x_2(t)^2 + \dots + x_{n_quad}(t)^2]$

Outputs:

Column vector with the following scalar values :

Call option price = $P_{T_0,T_1} * \text{MAX}(P_{T_1,T_2} - K, 0)$

P_{T_0,T_1} = price of bond maturing at T1, viewed at T0 (note that $P_{T_1,T_1} = 1$)

P_{T_0,T_2} = price of bond maturing at T2, viewed at T0 (note that $P_{T_2,T_2} = 1$)

P_{T_1,T_2} = price of bond maturing at T2, viewed at T1

Simulated value for $r(T_1)$

Function P_QUAD(alpha, beta, sigma, n_quad, x0, t1, T2)

This function is an implementation of the separable quadratic multifactor model closed form bond price equations given by Jamshidian (Bond, futures and option evaluation in the quadratic interest rate model, Applied Mathematical Finance, Vol. 3, 1996, pp. 93-115). See in particular Section 8, pages 104 and 106. Note that alpha, beta and sigma are all assumed to be known constants and x0 is common for all x_i .

Inputs:

alpha	= Vasicek mean reversion interest rate level (decimal form %)
beta	= speed of mean reversion in the Vasicek model
sigma	= volatility of Vasicek short term interest rate (decimal form %)
n_quad	= number of Vasicek factors in the separable quadratic model
x0	= initial Vasicek short term interest rate (decimal form %)
t1	= time at which bond price is determined (years)
T2	= bond maturity (years)

Vasicek short rate dynamics : $dx_i(t) = (\alpha - \beta x_i(t))dt + \sigma dW(t)$

Separable quadratic model : $r(t) = 0.5[x_1(t)^2 + x_2(t)^2 + \dots + x_{n_quad}(t)^2]$

Output:

Scalar value

P_t1_T2 = price of a zero coupon bond maturing at T2, viewed at t1

5 VBA Program Module Listings

The following pages contain the two Main VBA modules with the functions described in the previous section. The listing for Rebel_NM2P_CIR_MC_Simulation.xls is given first followed by the code for Rebel_NM2P_QUAD_MC_Simulation.xls.


```
' Rebel_NM2P_CIR_MC_Simulation.xls
' Gerhard Rebel
```

Option Explicit

```
' -----
' Pricing of a call on a zero coupon bond where the bond matures at time T2 and
' the call option matures at time T1,  $T1 < T2$ . In the discrete framework, T1 is
' denoted by  $m*dt$  and T2 by  $n*dt$  where  $m < n$ .
' This implementation uses the CIR short rate dynamics as specified in
' Glasserman Page 120, Equation 3.62
' Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003
' The function returns the value of the call and the three zero coupon bond
' prices : P_T0_T1, P_T0_T2, P_T1_T2
' It also returns the expected value of the short rate at time T1,  $r(T1) = r(m)$ 
' -----
```

Function Call_on_zero_bond_CIR(r0, K, alpha, b, sigma, n, m, dt, num_sims)

```
    Dim i As Integer ' loop counter, time steps

    Dim j As Integer ' loop counter, Monte Carlo simulations

    Dim i_count ' loop counter, for simulation of  $r(t)$ , see Glasserman Page 133

    Dim P_T0_T1_sum, P_T0_T2_sum, P_T1_T2_sum ' summation of all the simulated bond prices

    Dim r_val_T1_sum ' summation of all the simulated  $r(m) = r(T1)$  values

    Dim exp_T0_T1, exp_T1_T2 ' note that  $\exp_{T0_T2} = \exp_{T0_T1} + \exp_{T1_T2}$ 

    Dim P_T0_T1, P_T0_T2, P_T1_T2 ' expected values for all bond price simulations

    Dim r_val_T1 ' expected value for all the  $r(m) = r(T1)$  simulations

    Dim Z ' std normal random variable

    Dim d ' see Glasserman Page 133

    ' check that  $m < n$  i.e.  $T1 < T2$ 

    If m >= n Then

        m = n - 1

        MsgBox ("Note: m has been adjusted automatically to  $n - 1 =$ " & m)

    End If

    ReDim r_val(0 To n - 1) ' simulated short rate values

    ReDim Output_array(0 To 4) ' output data array

    ' Initialise the simulation variables

    r_val_T1_sum = 0
```

```

P_T0_T1_sum = 0
P_T0_T2_sum = 0
P_T1_T2_sum = 0
d = Int(4 * b * alpha / sigma ^ 2) ' See Glasserman Page 133
' Check relationship between alpha b and sigma
If ((d - (4 * b * alpha / sigma ^ 2)) <> 0) Then MsgBox ("Note: d is not an integer")
If ((2 * alpha * b - sigma ^ 2) <= 0) Then MsgBox ("Note: 2 * alpha * b - sigma ^ 2 should be greater than zero")
' Run the Monte Carlo simulation, j are the individual simulations
For j = 1 To num_sims
    r_val(0) = r0
    exp_T0_T1 = r0 ' because the loop below starts at i = 1 and we need the sum to include values from i = 0
    exp_T1_T2 = 0
    ' i are the time steps for one simulation path
    For i = 1 To n - 1 ' we only need the to calculate up to the second last time point
        ' -----
        ' Simulation of r(t), see Glasserman Page 133
        ' Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003
        r_val(i) = 0
        For i_count = 1 To d
            Z = Application.NormSInv(Rnd) ' std normal random variable with mean = 0 and std dev = 1
            r_val(i) = r_val(i) + _
            (Exp(-0.5 * alpha * dt) * (r_val(i - 1) / d) ^ 0.5 + sigma / 2 * (1 / alpha * (1 - Exp(-alpha * dt))) ^ 0.5 * Z) ^ 2
        Next i_count
        ' -----
        ' Alternative Simulation of r(t), see Figure 3.5 of Glasserman, Page 124
        ' Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003
        ' r_val(i) = CIR_rtipl(alpha, b, r_val(i - 1), i, dt, sigma, d)
        ' -----
    End For
    If i < m Then

```

```

        exp_T0_T1 = exp_T0_T1 + r_val(i) ' summation of all the short rates from i = 0 to m-1
    Else
        exp_T1_T2 = exp_T1_T2 + r_val(i) ' summation of all the short rates from i = m up to n-1
    End If

Next i

r_val_T1_sum = r_val_T1_sum + r_val(m) ' keep track of the value of r_val(m) = r(T1)

P_T0_T1_sum = P_T0_T1_sum + Exp(-dt * exp_T0_T1) ' summation of all the simulated P_T0_T1 bond prices
P_T0_T2_sum = P_T0_T2_sum + Exp(-dt * (exp_T0_T1 + exp_T1_T2)) ' summation of all the simulated P_T0_T2 bond prices
P_T1_T2_sum = P_T1_T2_sum + Exp(-dt * exp_T1_T2) ' summation of all the simulated P_T1_T2 bond prices

Next j

' Calculate the bond prices and the short rate at T1
r_val_T1 = r_val_T1_sum / num_sims ' expected value for all simulations, r(T1)
P_T0_T1 = P_T0_T1_sum / num_sims ' expected value for all simulations, bond maturing T1, viewed at T0
P_T0_T2 = P_T0_T2_sum / num_sims ' expected value for all simulations, bond maturing T2, viewed at T0
' P_T0_T2 is not needed for the option price calculation

P_T1_T2 = P_T1_T2_sum / num_sims ' expected value for all simulations, bond maturing T2, viewed at T1

' Calculate the call option price and create the output array

Output_array(0) = P_T0_T1 * Application.Max(P_T1_T2 - K, 0) ' discounted terminal value
Output_array(1) = P_T0_T1
Output_array(2) = P_T0_T2
Output_array(3) = P_T1_T2
Output_array(4) = r_val_T1

Call_on_zero_bond_CIR = Application.Transpose(Output_array)

End Function ' Call_on_zero_bond_CIR(...)

' -----
' Implementation of Figure 3.5 of Glasserman, Page 124
' Simulation of r(t) under CIR short rate dynamics
' Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003
' -----

Function CIR_rtipl(alpha, b, rti, i, dt, sigma, d)

    Dim n, Z, X, c, lamda

    ' Note that d = 4 * b * alpha / sigma ^ 2

```

```

If d > 1 Then

    c = sigma ^ 2 * (1 - Exp(-alpha * dt)) / (4 * alpha)

    lamda = rti * Exp(-alpha * dt) / c

    Z = Application.NormSInv(Rnd) ' std normal random variable with mean = 0 and std dev = 1

    X = Application.ChiInv(Rnd, d - 1) ' returns the inverse of the one-tailed probability of the chi-squared distribution

    CIR_rtip1 = c * ((Z + lamda ^ 0.5) ^ 2 + X)

ElseIf d <= 1 Then

    c = sigma ^ 2 * (1 - Exp(-alpha * dt)) / (4 * alpha)

    lamda = rti * Exp(-alpha * dt) / c

    n = Application.Poisson(i, lamda / 2, False) ' returns the Poisson probability mass function, ** not sure about the i ?? **

    X = Application.ChiInv(Rnd, d + 2 * n) ' returns the inverse of the one-tailed probability of the chi-squared distribution

    CIR_rtip1 = c * X

Else

    CIR_rtip1 = rti

End If

End Function ' CIR_rtip1(...)

' -----
' Implementation of CIR closed form bond price equations
' See Hull Page 542-543
' Options, Futures, and Other Derivatives, Fifth Edition 2003
' -----

Function P_CIR(alpha, b, rt1, sigma, t1, T2)

    Dim gamma, B_t1_T2, A_t1_T2

    gamma = (alpha ^ 2 + 2 * sigma ^ 2) ^ 0.5

    B_t1_T2 = 2 * (Exp(gamma * (T2 - t1)) - 1) / ((gamma + alpha) * (Exp(gamma * (T2 - t1)) - 1) + 2 * gamma)

    A_t1_T2 = ((2 * gamma * Exp((alpha + gamma) * (T2 - t1) / 2)) /
        ((gamma + alpha) * (Exp(gamma * (T2 - t1)) - 1) + 2 * gamma)) ^ (2 * alpha * b / sigma ^ 2)

    P_CIR = A_t1_T2 * Exp(-B_t1_T2 * rt1)

End Function ' P_CIR(...)

```

```
' Rebel_NM2P_QUAD_MC_Simulation.xls
' Gerhard Rebel
```

Option Explicit

```
' -----
' Pricing of a call on a zero coupon bond where the bond matures at time T2 and
' the call option matures at time T1, T1 < T2. In the discrete framework, T1 is
' denoted by m*dt and T2 by n*dt where m < n.
' Separable quadratic model (with Vasicek short rate dynamics) see Jamshidian Pages 104-106
' Bond, futures and option evaluation in the quadratic interest rate model,
' Applied Mathematical Finance, Vol. 3, 1996
'
' This implementation uses the Vasicek short rate dynamics as modified from
' Glasserman Pages 108-111, Equations 3.39 and 3.46 to form factors in the
' quadratic model
' Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003
' The function returns the value of the call and the three zero coupon bond
' prices : P_T0_T1, P_T0_T2, P_T1_T2
' It also returns the expected value of the short rate at time T1, r(T1) = r(m)
' -----
```

Function Call_on_zero_bond_QUAD(x0, K, alpha, beta, sigma, n, m, dt, num_sims, n_quad)

```
    Dim i As Integer ' loop counter for time steps

    Dim j As Integer ' loop counter for Monte Carlo simulations

    Dim n_quad_count ' loop counter for factors in the separable quadratic model

    Dim P_T0_T1_sum, P_T0_T2_sum, P_T1_T2_sum ' summation of all the simulated bond prices

    Dim r_val_T1_sum ' summation of all the simulated short rate r(m) = r(T1) values

    Dim exp_T0_T1, exp_T1_T2 ' note that exp_T0_T2 = exp_T0_T1 + exp_T1_T2

    Dim P_T0_T1, P_T0_T2, P_T1_T2 ' expected values for all bond price simulations

    Dim r_val_T1 ' expected value for all the short rate r(m) = r(T1) simulations

    Dim Z ' std normal random variable

    ' check that m < n i.e. T1 < T2

    If m >= n Then

        m = n - 1

        MsgBox ("Note: m has been adjusted automatically to n - 1 = " & m)

    End If

    ReDim r_val(0 To n - 1) ' simulated quadratic short rate values

    ReDim x_val(0 To n - 1, 1 To n_quad) ' simulated Vasicek short rate values used to make up r_val
```

```

ReDim Output_array(0 To 4) ' output data array

' Initialise the simulation variables

r_val_T1_sum = 0
P_T0_T1_sum = 0
P_T0_T2_sum = 0
P_T1_T2_sum = 0

' Run the Monte Carlo simulation, j are the individual simulations
For j = 1 To num_sims

    r_val(0) = 0 ' initial value for the loop

    ' Determine the value of r(0) given x0 and the number of factors, n_quad

    For n_quad_count = 1 To n_quad

        x_val(0, n_quad_count) = x0 ' given constant value

        r_val(0) = r_val(0) + 0.5 * x_val(0, n_quad_count) ^ 2 ' see Jamshidian Page 104, Section 8

    Next n_quad_count

    exp_T0_T1 = r_val(0) ' because the loop below starts at i = 1 and we need the sum to include values from i = 0
    exp_T1_T2 = 0 ' we will start adding to this from i = m

    ' i are the time steps for one simulation path

    For i = 1 To n - 1 ' we only need the to calculate up to the second last time point

        ' Simulation of r(t), see Glasserman Page 110, Equation 3.46 for the Vasicek short rate dynamics
        ' Monte Carlo Methods in Financial Engineering, Springer Verlag, July 2003

        r_val(i) = 0 ' initial value for the loop

        For n_quad_count = 1 To n_quad ' i.e. factors in the separable quadratic model

            Z = Application.NormSInv(Rnd) ' std normal random variable with mean = 0 and std dev = 1

            x_val(i, n_quad_count) = Exp(-beta * dt) * x_val(i - 1, n_quad_count) + _
                alpha / beta * (1 - Exp(-beta * dt)) + _
                sigma * (1 / (2 * beta) * (1 - Exp(-2 * beta * dt))) ^ 0.5 * Z

            r_val(i) = r_val(i) + 0.5 * x_val(i, n_quad_count) ^ 2 ' see Jamshidian Page 104, Section 8

        Next n_quad_count

        If i < m Then

            exp_T0_T1 = exp_T0_T1 + r_val(i) ' summation of all the short rates from i = 0 to m-1

```

```

Else
    exp_T1_T2 = exp_T1_T2 + r_val(i) ' summation of all the short rates from i = m up to n-1
End If

Next i

r_val_T1_sum = r_val_T1_sum + r_val(m) ' keep track of the value of r_val(m) = r(T1)

P_T0_T1_sum = P_T0_T1_sum + Exp(-dt * exp_T0_T1) ' summation of all the simulated P_T0_T1 bond prices

P_T0_T2_sum = P_T0_T2_sum + Exp(-dt * (exp_T0_T1 + exp_T1_T2)) ' summation of all the simulated P_T0_T2 bond prices

P_T1_T2_sum = P_T1_T2_sum + Exp(-dt * exp_T1_T2) ' summation of all the simulated P_T1_T2 bond prices

Next j

' Calculate the expected bond prices and the expected short rate at T1

r_val_T1 = r_val_T1_sum / num_sims ' expected value for all simulations, r(T1) = r(m)

P_T0_T1 = P_T0_T1_sum / num_sims ' expected value for all simulations, bond maturing T1, viewed at T0

P_T0_T2 = P_T0_T2_sum / num_sims ' expected value for all simulations, bond maturing T2, viewed at T0
' P_T0_T2 is not needed for the option price calculation

P_T1_T2 = P_T1_T2_sum / num_sims ' expected value for all simulations, bond maturing T2, viewed at T1

' Calculate the call option price and create the output array

Output_array(0) = P_T0_T1 * Application.Max(P_T1_T2 - K, 0) ' discounted terminal value of call option
Output_array(1) = P_T0_T1 ' note that P_T0_T1 and P_T1_T2 are already expected values
Output_array(2) = P_T0_T2
Output_array(3) = P_T1_T2
Output_array(4) = r_val_T1

Call_on_zero_bond_QUAD = Application.Transpose(Output_array) ' create a column vector for output to XLS sheet

End Function ' Call_on_zero_bond_CIR(...)

' -----
' Implementation of the separable quadratic model bond price equations
' See Jamshidian, Bond, futures and option evaluation in the quadratic interest
' rate model, Applied Mathematical Finance, Vol. 3, 1996, pp. 93-115.
' See in particular, Section 8, Pages 104 and 106
' Note that alpha, beta and sigma are all assumed to be known constants
' x0 is common for all xi
' n_quad is the number of factors in the separable quadratic model
' -----

Function P_QUAD(alpha, beta, sigma, n_quad, x0, t1, T2)

    Dim gamma, tau, h_tau, B_t1_T2, small_B_t1_T2, small_C_t1_T2, x_t1, x_t1_sqrd

```

```

gamma = (sigma ^ 2 + beta ^ 2) ^ 0.5

tau = T2 - t1

' *****
' Note that the Jamshidian equation for h_tau has "... + gamma + beta)" which is incorrect,
' it should be "... + gamma - beta)"
' Compare with James / Webber, Equation 9.68 which is Jamshidian's B_t1_T2 containing h_tau
' *****

h_tau = 1 / ((gamma + beta) * Exp(2 * gamma * tau) + gamma - beta)

B_t1_T2 = h_tau * (Exp(2 * gamma * tau) - 1)

small_B_t1_T2 = alpha * h_tau * (Exp(gamma * tau) - 1) ^ 2 / gamma

small_C_t1_T2 = n_quad * (0.5 * ((alpha ^ 2 / gamma ^ 2) - (sigma ^ 2 / (gamma - beta))) * tau _
- 0.5 * Log(2 * gamma * h_tau) _
+ alpha ^ 2 / gamma ^ 3 / (gamma + beta) * _
(h_tau * (2 * beta * (gamma + beta) * Exp(gamma * tau) + sigma ^ 2 - beta ^ 2) - beta - gamma / 2))

' Note that x_t1 and x_t1_sqaured are expected values, E[], used for the closed form bond price calculations
' These are based on the E[] and VAR[] of the Vasicek short rate process given by Glasserman and others
' E[x(t)^2] = (E[x(t)])^2 + VAR[x(t)] is used to determine x_t1_sqaured, based on x_t1
' dxi(t) = ( alpha - beta*xi(t) ) *dt + sigma*dW(t)

x_t1 = x0 * Exp(-beta * t1) + alpha / beta * (1 - Exp(-beta * t1)) ' see Glasserman Page 110, bottom of page for E[x(t)]

x_t1_sqrd = (x_t1) ^ 2 + sigma ^ 2 / (2 * beta) * (1 - Exp(-2 * beta * t1)) ' see Glasserman Page 111, top of page for VAR[x(t)]

P_QUAD = Exp(-n_quad * (0.5 * B_t1_T2 * x_t1_sqrd + small_B_t1_T2 * x_t1) - small_C_t1_T2) ' Jamshidian Page 104, Section 8

End Function ' P_QUAD(...)

```

6 Project Brief and Issued Literature

The attached pages contain the project brief, as issued, along with the referenced literature.

5

Quadratic Interest Rate Model simulation

Jamshidian *Bond, futures and option evaluation in the quadratic interest rate model*
Applied Mathematical Finance 1996 pp 93-115

Glasserman book *Monte Carlo Methods in Financial Engineering*
pp 124, 132-133, and other
ISMA Centre just bought a copy for use by students

James / Webber book *Interest Rate Modelling* Ch 9.3.2 pp 229-231

Simulate the short term rate r
Compute the corresponding bond price
Compute the price of a European call option on the simulated bond price
Compare the simulated call price with the closed form formulas given by Jamshidian

assigned to student:

G. Rebel

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Applied Mathematical Finance 3, 93–115 (1996)

Bond, futures and option evaluation in the quadratic interest rate model

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Revised February 1996

This paper develops the quadratic interest-rate model of Beaglehole and Tenney in detail. For the quadratic model as well as the multifactor Cox–Ingersoll–Ross square-root model, explicit pricing formulae in terms of one-dimensional integrals of elementary functions are given for bond options, bond exchange options, caps, options on bond futures and forward contracts, and futures delivery options. For the quadratic model, certain forward and transport equations are found that explicitly determine the dynamics of the term structure in terms of initial yield and volatility curves. These option-pricing formulae are thus determined in terms of the initial curves. Some shortcomings of the model are identified. New formulae for some distributions and their truncated moments are also derived.

Keywords: principal value integral, noncentral chi-squared distribution, forward risk adjustment, forward and transport equations, yield curve calibration

1. Introduction

The multifactor quadratic interest-rate model was introduced by Beaglehole and Tenney (1991). It was further developed in El Karoui *et al.* (1992). This paper derives a more complete list of its properties, especially for the special case of 'multifactor separable' quadratic models, i.e. models which are essentially a sum of independent one-factor models. The new results include analytical calibration of the yield curve by algebraic formulae, 'transport equations' that express the coefficients of forward interest rates at future dates as algebraic functions of their initial value, formulae for the distribution of interest rates, pricing formulae for futures, zero-coupon bond options and other European derivatives in terms of one-dimensional integrals of elementary functions and the initial term structure and, for the constant coefficient case, algebraic formulae for prices of zero-coupon bonds. We also point out some anomalies of the quadratic interest-rate model.

In the quadratic interest-rate model, the spot interest rate is a quadratic function of a (multivariate) Gaussian state variable. For the constant coefficient case, the quadratic model is closely related to affine yield models studied by Duffie and Kan (1992) and its special cases such as the one-factor square-root model of Cox *et al.* (1981, 1985) and their multifactor generalization in Chen and Scott (1992) and Longstaff and Schwartz (1992). They are linked by similar formulae for zero-coupon bonds deduced from Riccati equations, and the fact that interest rates in both classes of models are chi-squared distributed.

Here we are primarily interested in the case of time-varying coefficients, so that the model can be calibrated to an initial yield curve. Such quadratic models resemble the 'simple square-root models'

introduced in Jamshidian (1995). But, as we shall see, there are in general important differences which argue in favour of simple square-root models. An interesting connection between the two classes of model has been established by Maghsoodi (1996), who shows that certain multifactor quadratic models are equivalent to one-factor simple square-root models with 'integer dimension'.

The paper is organized as follows. Section 2 provides formulae for the distribution function and truncated moments of linear combinations of independent non-central chi-squared distributions as one-dimensional integrals of elementary functions. These formulae are useful for computational purposes even in the single-factor case. The derivation uses known formulae for the Fourier transforms of the Heaviside function and its derivative as generalized functions. This technique more generally furnishes the distribution function and truncated moments of any distribution as the integral of an elementary function, provided the characteristic function of the distribution has an elementary form. These results are of independent interest.

Section 3 presents the basic properties of forward risk adjustment and discusses its origins. The usefulness of this concept for deriving option-pricing formulae is well known. We will use it here also to derive the transport equations and other key structural formulae, such as the zeroth order coefficient of bond prices (or forward rates).

Section 4 applies the results of Sections 2 and 3 to derive various option-pricing formulae which are applicable to multifactor models with (generalized) non-central chi-squared distributed interest rates. These include separable multifactor Cox–Ingersoll–Ross (CIR) and quadratic models. The pricing formulae are for zero-coupon bond options, options to exchange two (zero-coupon) bonds, options on bond forward contracts, caps, options on bond futures and the delivery option in bond futures. Again, they are all expressed explicitly as one-dimensional integrals of elementary functions. These results are applied to the multifactor CIR model in Section 5.

After some preliminaries in Section 6, Section 7 discusses the general multifactor quadratic interest-rate model, and presents the system of ordinary differential equations (ODE) that zero-coupon bond prices satisfy. Section 8 presents detailed properties of separable multifactor quadratic models, including distributional characteristics and transport and forward equations. This section also provides bond- and futures-pricing formulae for the constant coefficient case. An examination of these formulae reveals that anomalous behaviour results when the underlying Gaussian state variable lies within a certain range, i.e. in the one-factor case, long-term rates will be negatively correlated with short-term rates at these states. This and other undesirable features of the quadratic model stem from the fact that its spot interest rate fails to be a Markov (diffusion) process.

Section 9 applies the results of Section 8 to the problem of fitting an initial yield curve (and if desired also a volatility curve). Further relevant structural formulae of the quadratic model are recorded there. The calibration is entirely analytic in the main case of interest, and in all other cases at most a single ODE needs to be solved. Section 10 concludes that simple square-root models are better models than quadratic interest-rate models. An appendix contains proofs of Theorems 2.1, 7.1 and 7.2, and some extensions.

2. Linear combinations of independent non-central chi-squared distributions

In the multifactor models studied in this paper, interest rates turn out to be distributed as a linear

combination of independent non-central chi-squared random variables, and bond prices will be exponentials of such distributions. In this section, we derive some formulae for these distributions and their truncated first moment in terms of a one-dimensional integral of elementary functions which will later furnish option-pricing formulae.

If Y is a random variable such that $Y = \alpha + \beta_1 X_1 + \dots + \beta_n X_n$ for some scalars α, β_i and some independent random variables X_i , each non-centrally chi-squared distributed with $\nu_i > 0$ degrees of freedom and a non-centrality parameter $\lambda_i \geq 0$, then we write $Y \sim \chi_n^2(\nu, \lambda, \beta, \alpha)$ ($\nu = (\nu_1, \dots, \nu_n)$, etc.).¹ From the formula for the moment generating function of X_i , we have

$$\Psi_n(\nu, \lambda, \beta, \alpha) \equiv E[e^{-Y}] = \exp\left(-\sum_{i=1}^n \left(\frac{1}{2}\nu_i \log(1 + 2\beta_i) + \beta_i \lambda_i / (1 + 2\beta_i)\right) - \alpha\right) \quad (\beta_i > -\frac{1}{2})$$

Note that $E[Y] = \alpha + \sum \beta_i(\nu_i + \lambda_i)$. The characteristic function of Y admits a similar algebraic formula. Inverting it by the inverse Fourier transform, we obtain a formula for the density function as an integral of an elementary function. It is interesting that the distribution function of Y can also be represented as a one-dimensional integral:

$$\chi_n^2(y; \nu, \lambda, \beta) \equiv \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Psi_n\left(\frac{\nu}{2}, \lambda, 2\xi^2 \beta^2, 0\right) \sin(\xi y - \Phi_n(\nu, \lambda, \xi \beta)) \frac{d\xi}{\xi}$$

where ν, λ, β , are n -dimensional vectors, and y is a scalar, and

$$\Phi_n(\nu, \lambda, \gamma) \equiv \sum_{i=1}^n \left(\frac{\nu_i}{2} \arctan(2\gamma_i) + \frac{\gamma_i \lambda_i}{1 + 4\gamma_i^2} \right)$$

Another integral of interest is²

$$C_n(y; \nu, \lambda, \beta) \equiv \frac{1}{\pi} \int_0^\infty \left[1 - \Psi_n\left(\frac{\nu}{2}, \lambda, 2\xi^2 \beta^2, 0\right) \cos(\xi y - \Phi_n(\nu, \lambda, \xi \beta)) \right] \frac{d\xi}{\xi^2}$$

The following result, proved in the appendix, summarizes the formulae relevant to European option evaluation.

Theorem 2.1: If $Y \sim \chi_n^2(\nu, \lambda, \beta, \alpha)$, $Y' \sim \chi_n^2(\nu, \lambda, \beta', \alpha')$, and $y \in \mathbb{R}$, the following hold:

$$\text{prob}[Y \leq y] = \chi_n^2(y - \alpha; \nu, \lambda, \beta)$$

$$E[(Y - y)_+] = \frac{1}{2}(E[Y] - y) + C_n(y - \alpha; \nu, \lambda, \beta)$$

$$E[(e^{-Y} - e^{-y})_+] = E[e^{-Y}] \chi_n^2\left(y - \alpha; \nu, \frac{\lambda}{1 + 2\beta}, \frac{\beta}{1 + 2\beta}\right) - e^{-y} \chi_n^2(y - \alpha; \nu, \lambda, \beta)$$

¹ Throughout the paper we use vector notation where the product and division of two vectors, like their sum and difference, is defined pointwise (also, $x_+ \equiv \max(x, 0)$).

² For numerical integration of these two integrals, it is useful to note that the (limiting) values of the two integrands at $\xi = 0$ are respectively $y - \beta(\nu + \lambda)$ and $\beta^2(\nu + 2\lambda) + (y - \beta(\nu + \lambda))^2/2$. Also note that both integrands are absolutely integrable.

$$\begin{aligned}
E[(e^{-Y} - e^{-Y'})_+] &= E[e^{-Y}] \chi_n^2 \left(\alpha' - \alpha; \nu, \frac{\lambda}{1+2\beta}, \frac{\beta - \beta'}{1+2\beta} \right) \\
&\quad - E[e^{-Y'}] \chi_n^2 \left(\alpha' - \alpha; \nu, \frac{\lambda}{1+2\beta'}, \frac{\beta - \beta'}{1+2\beta'} \right) \\
E[\min(e^{-Y}, e^{-Y'})] &= E[e^{-Y}] \chi_n^2 \left(\alpha - \alpha'; \nu, \frac{\lambda}{1+2\beta}, \frac{\beta' - \beta}{1+2\beta} \right) \\
&\quad + E[e^{-Y'}] \chi_n^2 \left(\alpha' - \alpha; \nu, \frac{\lambda}{1+2\beta'}, \frac{\beta - \beta'}{1+2\beta'} \right)
\end{aligned}$$

The first formula of the theorem gives the previously stated distribution of Y . The second and the third equations relate respectively to options on yields and options on zero-coupon bonds. The fourth and fifth formulae correspond more generally to an option to exchange two zero-coupon bonds, and are useful for the delivery option in bond futures contracts.

3. Forward risk adjustment

As mentioned in the introduction, forward risk adjustment will be used both to derive option prices and some of the structural equations of the quadratic model. To our knowledge, the concept was first introduced in Jamshidian (1987), where three of its basic properties were established for the general one-factor diffusion model, and applied to the Cox–Ingersoll–Ross square-root model and the Gaussian model with time-varying drift term. Evidently independently, El Karoui and Rochet (1989) presented and studied this concept. Reportedly, Geman (1989) also arrived at this concept. Some of the ideas of forward risk adjustment date back to Merton (1973), where variables were changed to forward prices to simplify the fundamental differential equation.

In Jamshidian (1987), as was customary then (e.g. as with risk-neutrality adjustment), instead of changing the measure, more old-fashionedly, but equivalently, the drift of processes was changed. Using Ito's division rule, it was shown that (i) the T -maturity forward-risk-adjusted drift is obtained by adding to the risk-neutral drift the volatility of the T -maturity zero-coupon bond times the process volatility; (ii) price of a contingent claim is the discount factor times the expected payoff of the forward-risk-adjusted process; and (iii) the forward interest rate is the expected value of forward-risk-adjusted spot interest rate.³ In the more modern language of change of measure, these properties can be restated as (i) the forward-risk-adjusted Brownian motion differs from the risk-neutral Brownian motion by the zero-coupon bond price volatility; (ii) forward asset prices are martingales in the forward-risk-adjusted measure; and (iii) forward interest rates are martingales in the forward-risk-adjusted measure.

³ As noted on the previous page these properties show 'discount bond price volatility plays the role of a market price of risk ... forward rates ... turn out to be expectation of the forward risk-adjusted interest rate process ... prices of contingent are discounted expected payoff values with respect to this [forward-risk-adjusted] process ... [forward risk adjustment] clarifies the relation between the one-factor term structure model and Merton's option model of 1973 which is essentially in terms of forward prices.'

Forward risk adjustment can be thought of as taking the zero-coupon bond as the 'numeraire'. Of the three basic properties, the first two are essentially shared by any change of numeraire. But, the third is specific to forward risk adjustment. Further properties specific to forward risk adjustment were found by El Karoui *et al.* (1992) and applied to the quadratic model.

Let $r(t)$ denote the spot interest rate, and assume that the model is arbitrage free so that T -maturity zero-coupon bond prices and forward interest rates are given respectively by

$$P_T(t) \equiv E_t \left[\exp \left(- \int_t^T r(s) ds \right) \right] \quad (0 \leq t \leq T)$$

$$r_T(t) \equiv -\partial \log (P_T(t)) / \partial T$$

$$= E_t \left[r(T) \exp \left(- \int_t^T r(s) ds \right) \right] / P_T(t)$$

$$(r(t) = r_t(t))$$

where $E_t[\cdot]$ denotes the risk-neutral expectation operator conditioned on time t . Assume further that $P_T(t)$ are Ito processes in the risk-neutral measure, so that

$$dP_T(t)/P_T(t) = r(t)dt - \sigma_T(t) dz(t)$$

$$(z(t) = (z_1(t), \dots, z_n(t))')$$

where $z(t)$ is a standard n -dimensional Brownian motion in the risk-neutral measure, and (row vector) $\sigma_T(t)$ is the (in general stochastic) bond-price volatility ($\sigma_t(t) = 0$)⁴. Then, for each $T > 0$, by Girsanov's theorem, there is an equivalent measure under which the process

$$z^T(t) \equiv z(t) + \int_0^t \sigma_T(s)' ds \quad (dz^T(t) = dz(t) + \sigma_T(t)' dt)$$

is a Brownian motion on $[0, T]$.⁵ Denoting its conditional expectation operator by $E_t^T[\cdot]$, using Ito's division rule or Girsanov's formula, for any Ito process $X(t)$ one easily obtains⁶

$$E_t^T[X(T)] = E_t[\exp \left(- \int_t^T r(s) ds \right) X(T)] / P_T(t) \quad (3.1)$$

⁴ So, if $\lambda(t)$ is the market price of risk, then $w(t) \equiv z(t) + \int_0^t \lambda(s) ds$ is a Brownian motion in the actual measure.

⁵ Its Radon–Nikodym derivative with respect to the risk-neutral measure is the random variable

$$\exp \left(- \int_0^T (\frac{1}{2} |\sigma_T(t)|^2 dt + \sigma_T(t) dz(t)) \right) = \exp \left(- \int_0^T r(t) dt \right) / P_T(0)$$

⁶ Indeed, integrating the equation $dP_s/P_s = r dt - \sigma_s dz$ gives

$$\exp \left(- \int_t^T (\frac{1}{2} |\sigma_s(u)|^2 du + \sigma_s(u) dz(u)) \right) = \exp \left(- \int_t^T r(u) du \right) P_s(T)/P_s(t) \quad (t \leq T \leq s)$$

Hence, by Girsanov's formula, we have more generally than (3.1),

$$E_t^s[X(T)] = E_t[X(T) \exp \left(- \int_t^T r(u) du \right) P_s(T)] / P_s(t) \quad (t \leq T \leq s)$$

In particular, we have $r_T(t) = E_t^T[r(T)]$, while for bond prices we obtain,

$$P_s(t) = P_T(t)E_t^T[P_s(T)] \quad (t \leq T \leq s) \quad (3.2)$$

4. Option-pricing formulae for χ_n^2 -distributed interest-rate models

Combining the results of the previous two sections, we obtain option-pricing formulae for models in which interest rates are distributed as a linear combination of independent chi-squared distributions. Equations (3.1) and (3.2) used in conjunction with Theorem 2.1 lead, without need for any further calculation, to pricing formulae for (i) zero-coupon bond options (and hence caps and floors); (ii) caplets on the spot rate $r(t)$ (which is related to American bond option valuation (see Jamshidian (1995))); (iii) options to exchange two zero-coupon bonds; (iv) zero-coupon bond futures; (v) options on zero-coupon bond futures; (vi) options on a zero-coupon bond forward contract; and (vii) the delivery (quality) option in a futures contract to deliver the cheapest of two (weighted) zero-coupon bonds. Below, if $X(t)$ is a process such that conditioned on time t the distribution of $X(T)$ in the risk-neutral (respectively forward-risk-adjusted) measure belongs to $\chi_n^2(\nu, \lambda, \beta, \alpha)$, we write,

$$X(T)|_t \sim \chi_n^2(\nu, \lambda, \alpha, \beta) \quad (\text{resp. } X(T)|_t^T \sim \chi_n^2(\nu, \lambda, \alpha, \beta))$$

Theorem 4.1: Let T denote an option expiration, $K > 0$ a strike price or rate, and 's-bond' denote the s -maturity zero-coupon bond, $s \geq T$.

- (i) Suppose $-\log(P_s(T))|_t^T \sim \chi_n^2(\nu_{T,s}(t), \lambda_{T,s}(t), \beta_{T,s}(t), \alpha_{T,s}(t))$ for some (\mathcal{F}_T -measurable random variables) $\nu_{T,s}(t), \lambda_{T,s}(t), \beta_{T,s}(t), \alpha_{T,s}(t)$. Then the price of a call option on the s -bond is

$$\begin{aligned} E_t \left[\exp \left(- \int_t^T r(u) du \right) (P_s(T) - K)_+ \right] \\ = P_s(t) \chi_n^2 \left(-\alpha_{T,s}(t) - \log K; \nu_{T,s}(t), \frac{\lambda_{T,s}(t)}{1 + 2\beta_{T,s}(t)}, \frac{\beta_{T,s}(t)}{1 + 2\beta_{T,s}(t)} \right) \\ - P_T(t) K \chi_n^2(-\alpha_{T,s}(t) - \log K; \nu_{T,s}(t), \lambda_{T,s}(t), \beta_{T,s}(t)) \end{aligned}$$

- (ii) Suppose $r(T)|_t^T \sim \chi_n^2(\nu_T(t), \lambda_T(t), \beta_T(t), \alpha_T(t))$ (for some $\nu_T(t), \lambda_T(t), \beta_T(t), \alpha_T(t)$). Then $r_T(t) = \alpha_T(t) + \Sigma \beta_T^i(t)(\nu_T^i(t) + \lambda_T^i(t))$, and the price of a caplet on the spot interest rate is

$$\begin{aligned} E_t \left[\exp \left(- \int_t^T r(u) du \right) (r(T) - K)_+ \right] = P_T(t) \left[\frac{1}{2} (r_T(t) - K) \right. \\ \left. + C_n(K - \alpha_T(t); \nu_T(t), \lambda_T(t), \beta_T(t)) \right] \end{aligned}$$

- (iii) Suppose $-\log(P_s(T))|_t^T \sim \chi_n^2(\nu_T(t), \lambda_T(t), \beta_{T,s}(t), \alpha_{T,s}(t))$, and similarly for $-\log(P_{s'}(T))$.

Then, the price of the exchange option to put κ' s' -bonds and call κ s -bonds is

$$\begin{aligned} E_t \left[\exp \left(- \int_t^T r(u) du \right) (\kappa P_s(T) - \kappa' P_{s'}(T))_+ \right] \\ = \kappa P_s(t) \chi_n^2 \left(\alpha_{T,s'}(t) + \log \frac{\kappa}{\kappa'} - \alpha_{T,s}(t); \nu_T(t), \frac{\lambda_T(t)}{1 + 2\beta_{T,s}(t)}, \frac{\beta_{T,s}(t) - \beta_{T,s'}(t)}{1 + 2\beta_{T,s}(t)} \right) \\ - \kappa' P_{s'}(t) \chi_n^2 \left(\alpha_{T,s'}(t) + \log \frac{\kappa}{\kappa'} - \alpha_{T,s}(t); \nu_T(t), \frac{\lambda_T(t)}{1 + 2\beta_{T,s'}(t)}, \frac{\beta_{T,s}(t) - \beta_{T,s'}(t)}{1 + 2\beta_{T,s'}(t)} \right) \end{aligned}$$

- (iv) Suppose $-\log(P_{s'}(T)/P_s(T))|_t^T \sim \chi_n^2(\nu_{T,s,s'}(t), \lambda_{T,s,s'}(t), \beta_{T,s,s'}(t), \alpha_{T,s,s'}(t))$. Then the price of a call option on the s -delivery forward contract on the s' -bond ($t \leq T \leq s \leq s'$) is⁷

$$\begin{aligned} E_t \left[\exp \left(- \int_t^T r(u) du \right) (P_{s'}(T) - K P_s(T))_+ \right] \\ = P_s(t) E_t^s \left[\left(\frac{P_{s'}(T)}{P_s(T)} - K \right)_+ \right] \\ = P_{s'}(t) \chi_n^2 \left(-\alpha_{T,s,s'}(t) - \log K; \nu_{T,s,s'}(t), \frac{\lambda_{T,s,s'}(t)}{1 + 2\beta_{T,s,s'}(t)}, \frac{\beta_{T,s,s'}(t)}{1 + 2\beta_{T,s,s'}(t)} \right) \\ - P_s(t) K \chi_n^2(-\alpha_{T,s,s'}(t) - \log K; \nu_{T,s,s'}(t), \lambda_{T,s,s'}(t), \beta_{T,s,s'}(t)) \end{aligned}$$

- (v) Suppose $-\log(P_{t^*}(s))|_t \sim \chi_n^2(\nu_{s,t^*}(t), \lambda_{s,t^*}(t), \beta_{s,t^*}(t), \alpha_{s,t^*}(t))$. Then the price of an s -delivery futures contract on the t^* -bond is⁸

$$F_{s,t^*}(t) \equiv E_t[P_{t^*}(s)] = \Psi_n(\nu_{s,t^*}(t), \lambda_{s,t^*}(t), \beta_{s,t^*}(t), \alpha_{s,t^*}(t))$$

- (vi) Let $F(t)$ be an Ito process such that $-\log(F(T))|_t^T \sim \chi_n^2(\nu_T(t), \lambda_T(t), \beta_T(t), \alpha_T(t))$. Then,

$$\begin{aligned} E_t \left[\exp \left(- \int_t^T r(u) du \right) (F(T) - K)_+ \right] = P_T(t) [\Psi_n(\nu_T(t), \lambda_T(t), \beta_T(t), \alpha_T(t)) \\ \times \chi_n^2(-\alpha_T(t) - \log K; \nu_T(t), \frac{\lambda_T(t)}{1 + 2\beta_T(t)}, \frac{\beta_T(t)}{1 + 2\beta_T(t)}) \\ - K \chi_n^2(-\alpha_T(t) - \log K; \nu_T(t), \lambda_T(t), \beta_T(t))] \end{aligned}$$

⁷ Evidently, the option to exchange two bonds as in part (iii) is equivalent to an option on a bond forward contract, as in part (iv). We thus have two different formulae for this option under two slightly different assumptions. In applications below both assumptions are met, providing consequently two different but equivalent formulae for this option.

⁸ We can show that in general

$$F_{T,s}(t) \equiv E_t[P_s(T)] = \frac{P_s(t)}{P_T(t)} E_t^T \left[\exp \left(- \int_t^T \text{cov}_u \left[\frac{dF_{T,s}(u)}{F_{T,s}(u)}, \frac{dP_T(u)}{P_T(u)} \right] du \right) \right] \quad (t \leq T \leq s)$$

In particular if $-\log(F_{s,t^*}(T))|_t^T \sim \chi_n^2(\nu_{T,s,t^*}(t), \lambda_{T,s,t^*}(t), \beta_{T,s,t^*}(t), \alpha_{T,s,t^*}(t))$, then the price of a European call option on the s -delivery futures on t^* -bond is given by the above formula.

(vii) Suppose $-\log(P_{t^*}(s))|_t \sim \chi_n^2(\nu_s(t), \lambda_s(t), \beta_{s,t^*}(t), \alpha_{s,t^*}(t))$, and similarly for $-\log(P_{t'}(s))$. Then the price of the futures contract to optionally deliver either $\kappa^* t^*$ -bonds or $\kappa' t'$ -bonds is

$$E_t[\min(\kappa^* P_{t^*}(s), \kappa' P_{t'}(s))] = \kappa^* F_{s,t^*}(t) \chi_n^2\left(\alpha_{s,t^*}(t) + \log \frac{\kappa'}{\kappa^*} - \alpha_{s,t'}(t); \nu_s(t), \frac{\lambda_s(t)}{1 + 2\beta_{s,t^*}(t)}, \frac{\beta_{s,t'}(t) - \beta_{s,t^*}(t)}{1 + 2\beta_{s,t^*}(t)}\right) + \kappa' F_{s,t'}(t) \chi_n^2\left(\alpha_{s,t'}(t) + \log \frac{\kappa^*}{\kappa'} - \alpha_{s,t^*}(t); \nu_s(t), \frac{\lambda_s(t)}{1 + 2\beta_{s,t'}(t)}, \frac{\beta_{s,t^*}(t) - \beta_{s,t'}(t)}{1 + 2\beta_{s,t'}(t)}\right)$$

5. Example: separable multifactor CIR square-root model

In this model, $r(t) = x_1(t) + \dots + x_n(t)$, where $x_i(t)$ follow independent diffusion processes:

$$dx_i = \kappa_i(\theta_i - x_i) dt + \sigma_i \sqrt{x_i} dz_i \quad (2\kappa_i\theta_i > \sigma_i^2)$$

For $n = 1$, this is the classical model of Cox *et al.* (1981). For the case of two factors, this model was studied by Chen and Scott (1992) and independently by Longstaff and Schwartz (1992) in a different but equivalent form. Their methods for deriving zero-coupon bond-option formulae lead in general to an n -dimensional integral of the chi-squared density. However, by the results of the previous section, we can write these as one-dimensional integrals. Chen and Scott (1992) also treat options on coupon bonds. For this case, our method does not provide an improvement over their formula.

Since $x_i(t)$ are independent bond prices and can be written

$$P_T(t) = \prod_{i=1}^n E_t \left[\exp \left(- \int_t^T x_i(u) du \right) \right] = \exp \left(- \sum_{i=1}^n B_i(T-t)x_i(t) - \alpha(T-t) \right)$$

for some functions $B_i(\tau)$ and $\alpha(\tau) (= \sum \kappa_i \theta_i \int_0^\tau B_i(s) ds)$ whose explicit forms are given by Cox *et al.* (1981, 1985). Using the well-known Laplace transform of the non-central chi-squared distribution, Jamshidian (1987) calculated the forward-risk-adjusted distribution of $r(t)$ in the one-factor case, from which it follows that

$$x_i(T) |_t^T \sim \chi_1^2 \left(\frac{4\kappa_i\theta_i}{\sigma_i^2}, \frac{4B_i'(T-t)}{\sigma_i^2 B_i(T-t)} x_i(t), \frac{1}{4} \sigma_i^2 B_i(T-t), 0 \right)$$

Hence (using vector notation),

$$-\log P_s(T) |_t^T \sim \chi_n^2 \left(\frac{4\kappa\theta}{\sigma^2}, \frac{4B'(T-t)}{\sigma^2 B(T-t)} x(t), \frac{1}{4} \sigma^2 B(T-t)B(s-T), \alpha(s-T) \right)$$

Theorems 4.1 (i), (ii), (iii) are therefore applicable, providing pricing formulae for bond options, caps on spot rate and bond-exchange options. According to Cox *et al.* (1981),

$$x_i(T) |_t \sim \chi_1^2 \left(\frac{4\kappa_i\theta_i}{\sigma_i^2}, \frac{4\kappa_i e^{-\kappa_i(T-t)} x_i(t)}{\sigma_i^2 (1 - e^{-\kappa_i(T-t)})}, \frac{\sigma_i^2 (1 - e^{-\kappa_i(T-t)})}{4\kappa_i}, 0 \right)$$

Theorem 4.1 (v) thus provides a futures pricing formula in a similar linear exponential form to bond prices, with coefficients more explicitly provided in CIR (1981). Moreover, parts (vi) and (vii) provide pricing formulae for an option on a zero-coupon bond futures and for the delivery option. For the single-factor case, Carr (1987) provides more strongly a pricing formula for futures on several deliverable coupon bonds.

At least for the single-factor case, Jamshidian (1995) has shown, that in the 'simple square root models', where the coefficients are time varying but subject to $\kappa_i(t)\theta_i(t)/\sigma_i^2(t)$ being constant, interest rates and $\log(P_s(T))$ are χ_n^2 distributed, with coefficients given explicitly in terms of initial yield and volatility curves. Hence, we have similar pricing formulae for this more general class of models. Further, in the one-factor case, the formulae extend to coupon bonds.

6. A subfamily Ω_n of the class χ_n^2

In the square-root model, the 'degree of freedom' ν of the chi-squared distribution can be any real number. But, in the quadratic model, interest rates are squared norms of Gaussian variates, hence the degree of freedom is an integer. In preparation for the quadratic model, we thus define a subfamily of χ_n^2 with $\nu = 1$. If $Y = \Sigma(\frac{1}{2} B_i X_i^2 + b_i X_i) + c$, where X_1, \dots, X_n are independent normally distributed random variables with means μ_i and variances V_i , we write $Y \sim \Omega_n(B, b, c, \mu, V)$.⁹ We then have,

$$Y \sim \Omega_n(B, b, c, \mu, V) \Rightarrow Y \sim \chi_n^2(1, \lambda, \beta, \alpha)$$

$$\lambda \equiv (\mu + b/B)^2/V, \quad \beta \equiv \frac{1}{2}VB, \quad \alpha \equiv c - \frac{1}{2} \sum_{i=1}^n b_i^2/B_i$$

$$\text{prob}[Y \leq y] = \omega_n(y - c; B, b, \mu, V), \quad \omega_n(z; B, b, \mu, V) \equiv \chi_n^2 \left(z + \frac{1}{2} \sum_{i=1}^n b_i^2/B_i; 1, \lambda, \beta \right)$$

Theorem 4.1 is applicable. For example if $-\log(P_s(T))|_t^T \sim \Omega_n(B_s(T), b_s(T), c_s(T), \mu_T(t), V_T(t))$ then the bond-option-pricing formulae of part (i) apply by simply replacing $\lambda_{T,s}(t)$, etc., in terms of $B_s(T)$, etc., using the above transformation. Similarly, part (ii) for spot-rate caps, and parts (iv) and (v) for options on forward and futures contracts all go through by simply applying the above transformation. However, parts (iii) and (v) for exchange and delivery options do not apply directly. Indeed, the assumption in (iii) that $\lambda_T(t)$ is independent of s is violated. This is easily remedied. If $Y \sim \Omega_n(B, b, c, \mu, V)$ and $Y' \sim \Omega_n(B', b', c', \mu', V')$, then, similarly to its counterpart

⁹ Clearly, $\Omega_n(B, b, c, \mu, V) = \Omega_n(BV, (B\mu + b)\sqrt{V}, c + \Sigma b_i \mu_i + B_i \mu_i^2/2, 1)$.

for χ_n^2 in Theorem 2.1, we have,

$$E[(e^{-Y} - e^{-Y'})_+] = E[e^{-Y}] \Omega_n \left(c' - c; B - B', b - b', \frac{\mu - bV}{1 + BV}, \frac{V}{1 + BV} \right) \\ - E[e^{-Y'}] \Omega_n \left(c' - c; B - B', b - b', \frac{\mu - b'V}{1 + B'V}, \frac{V}{1 + B'V} \right)$$

Thus if $-\log(P_s(T))|_t^T \sim \Omega_n(B_s(T), b_s(T), c_s(T), \mu_T(t), V_T(t))$ and similarly for $-\log(P_{s'}(T))$, then, by forward risk adjustment, we obtain for the exchange option immediately,

$$E_t \left[\exp \left(- \int_t^T r(u) du \right) (\kappa P_s(T) - \kappa' P_{s'}(T))_+ \right] \\ = \kappa P_s(T) \omega_n \left(c_{s'}(T) + \log \frac{\kappa}{\kappa'} - c_s(T); B_s(T) - B_{s'}(T), b_s(T) \right. \\ \left. - b_{s'}(T), \frac{\mu_T(t) - b_s(T)V_T(t)}{1 + B_s(T)V_T(t)}, \frac{V_T(t)}{1 + B_s(T)V_T(t)} \right) \\ - \kappa' P_{s'}(T) \omega_n \left(c_{s'}(T) + \log \frac{\kappa}{\kappa'} - c_s(T); B_s(T) - B_{s'}(T), b_s(T) - b_{s'}(T), \right. \\ \left. \frac{\mu_T(t) - b_{s'}(T)V_T(t)}{1 + B_{s'}(T)V_T(t)}, \frac{V_T(t)}{1 + B_{s'}(T)V_T(t)} \right)$$

A similar formula is clear for the delivery option. For the next section it is useful to note

$$Y \sim \Omega_n(B, b, c, \beta x + a, V) \Rightarrow E[e^{-Y}] \\ = \exp \left(- \sum_{i=1}^n \left(\frac{1}{2} \frac{\beta_i^2 B_i}{1 + B_i V_i} x_i^2 + \frac{\beta_i(b_i + a_i B_i)}{1 + B_i V_i} x_i \right) - c - \bar{c}(B, b, a, V) \right) \quad (6:1) \\ \bar{c}(B, b, a, V) \equiv \sum_{i=1}^n \left(\frac{1}{2} \log(1 + V_i B_i) + \frac{B_i a_i^2 + 2b_i a_i - V_i b_i^2}{2(1 + V_i B_i)} \right)$$

7. The quadratic interest-rate model

In the quadratic interest-rate model, the spot interest rate is a quadratic function of a multivariate Gaussian state variable. By writing down the fundamental differential equation for zero-coupon bond prices, one finds that yields and forward rates of all maturities are also quadratic functions of the Gaussian state variable. It then follows that interest rates and logarithm of discount factors belong to the class of distribution Ω_n , and the results of the previous section are applicable.

In the general formulation of the quadratic model given by Beaglehole and Tenney (1991),

$$r(t) = \frac{1}{2} x(t)' Q(t) x(t) + g(t)' x(t) + f(t) \quad (x(t) = (x_1(t), \dots, x_m(t))')$$

where $Q(t)$ is a symmetric matrix, $g(t)$ is a vector, $f(t)$ a scalar, all smooth and deterministic, and $x(t)$ follows the Gaussian process

$$dx = (\alpha(t) - \beta(t)x) dt + \sigma(t) dz$$

for some smooth deterministic m -vector $\alpha(t)$, m by m matrix $\beta(t)$, and m by n matrix $\sigma(t)$. The fundamental partial differential equation for a contingent claim price $C(x, t)$ is

$$\frac{\partial C}{\partial t} + (\alpha - \beta x)' \frac{\partial C}{\partial x} + \frac{1}{2} tr \left(\sigma' \frac{\partial^2 C}{\partial x^2} \sigma \right) - rC = 0$$

In particular zero-coupon bond prices satisfy this equation, subject to the terminal condition of unity at maturity. We can solve the resulting partial differential equation by the separation of variable technique employed by Cox *et al.* (1981) for the square-root model. This results in a system of ordinary differential equations for the coefficients. Carrying this out, we find that the T -maturity zero-coupon bond price is given by

$$P_T(t) = \exp(-\frac{1}{2} x(t)' B_T(t) x(t) - b_T(t)' x(t) - c_T(t))$$

where, for each T , the matrix $B_T(t)$, vector $b_T(t)$ and scalar $c_T(t)$ functions of $t \leq T$ satisfy

$$d B_T / dt = \beta' B_T + B_T' \beta + B_T' \sigma \sigma' B_T - Q, \quad B_T(T) = 0$$

$$d b_T / dt - (\beta + \sigma \sigma' B_T)' b_T + B_T' \alpha + g = 0, \quad b_T(T) = 0$$

$$d c_T / dt + \alpha' b_T + \frac{1}{2} tr(\sigma' B_T \sigma) - \frac{1}{2} b_T' \sigma \sigma' b_T + f = 0, \quad c_T(T) = 0$$

Because the above (backward) system of ODEs for $B_T(t)$ is nonlinear, it may have a solution only for t near T . Nevertheless, since the right-hand side is symmetric, by (local) uniqueness, $B_T(t)$ will also be symmetric. In general, there is no closed-form solution for this system of ODE, even when the coefficients are constants.

Maghsoodi (1996) has proved the surprising result that certain n -factor quadratic models are equivalent to one-dimensional simple square-root models with integer 'dimension' $4\kappa(t)\theta(t)/\sigma^2(t) = n$: the spot rates of both models will have the same probability transition function. This holds for example when $Q(t)$ is the identity matrix, and $g(t) = f(t) = \alpha(t) = 0$. However, if $\alpha(t)$ is non-zero, this quadratic model cannot be reduced to the square-root model. For example, in the one factor case with $r(t) = x^2(t)$, the spot rate $r(t)$ will not even be a diffusion (Markov) process if the drift parameter $\alpha(t)$ is non-zero.

This can be seen by the following qualitative argument. Suppose $x(t)$ is strongly mean reverting relative to its volatility, and has a long-term expectation of, say, 3. Then, $r(t) = x^2(t)$ is expected to drift in the long term to (slightly more than) 9%. Suppose $r(t)$ is currently at 4%. Because of the large mean reversion, one would normally expect $r(t)$ to rise sharply. But, actually, it may fall sharply. Indeed, if $x(t) = -2$, then its expected direction is toward zero, which causes $r(t)$ to fall towards zero, before starting to rise to 9%. The expected direction of $r(t)$ therefore depends on the sign of $x(t)$, a quantity that is not determined by the level of $r(t)$ alone. This shows $r(t)$ cannot be a

Markovian process. To know the expected direction of $r(t)$, we have to keep a past record of whether $r(t)$ has 'bounced off' the origin $r = 0$ an odd or an even number of times.

This example also illustrates an undesirable feature of the quadratic model. Although the spot rate $r(t) = x^2(t)$ never becomes negative, it can become zero. But the parameters of the square-root model can be chosen so that its spot rate never reaches zero.

The multifactor quadratic model can be analysed in full generality, as demonstrated in El Karoui *et al.* (1992). But in order to derive a more explicit and complete set of properties, we will next specialize to 'separable multifactor' models.

8. Structure of the separable quadratic model

In this section we study in detail multifactor quadratic models which are essentially a sum of independent one-factor models. The analysis of these models reduces to that of the one-factor case, thus enabling sharp results. Yet these models allow non-perfectly correlated instantaneous changes in the yield curve, which is generally considered to be the main benefit of multifactor models. Specifically, we assume that $m = n$ and $\beta(t)$, $\sigma(t)$, and $Q(t)$ are diagonal matrices, and accordingly change notation to denote $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$, etc. By the linear transformation $x'_j = \sqrt{Q_j}x_j + g_j/\sqrt{Q_j}$, without loss of generality we assume that $Q_j(t) = 1$ and $g_j(t) = 0$. We may further assume that $f(t) = 0$, for otherwise $r_T(t)$ simply shifts by $f(T)$, without changing the analysis. So, we will henceforth assume the model

$$r(t) = \frac{1}{2}(x_1^2(t) + \dots + x_n^2(t))$$

$$dx_i = (\alpha_i(t) - \beta_i(t)x_i) dt + \sigma_i(t) dz_i$$

The T -maturity zero-coupon bond price and forward rate are now of the form¹⁰

$$P_T(t) = \exp\left(-\sum_{i=1}^n \left(\frac{1}{2} B_T^i(t) x_i^2(t) + b_T^i(t) x_i(t)\right) - c_T(t)\right)$$

$$r_T(t) = \sum_{i=1}^n \left(\frac{1}{2} B_T^i(t) x_i^2(t) + b_T^i(t) x_i(t)\right) + \dot{c}_T(t) \quad (\dot{c}_T(t) \equiv \frac{\partial c_T(t)}{\partial T} \text{ etc.})$$

In the T -maturity forward-risk-adjusted measure $x(t)$ is again Gaussian: in vector notation,

$$dx(t) = (\alpha(t) - \sigma^2(t)b_T(t) - (\beta(t) + \sigma^2(t)B_T(t))x(t)) dt + \sigma(t) dz^T(t)$$

It is now clear that interest rates are distributed Ω_n in both the risk-neutral and the forward-risk-adjusted measures, although we must still determine their parameters. We will first list the properties of the model in the following theorem (proved in the appendix), and then discuss their meaning and significance. Their application to fitting the yield curve will be discussed in the next section.

¹⁰ For a function of two variables written as $f_T(t)$, in this paper the dot notation signifies $\dot{f}_T(t) \equiv \partial f_T(t)/\partial T$, $\ddot{f}_T(t) \equiv \partial^2 f_T(t)/\partial T^2$, etc. Note, $f_s(T) = \partial f_s(T)/\partial s$, etc.

Theorem 8.1: For all $0 \leq t \leq u \leq T \leq s$ (using vector notation) the following equations hold:

$$\text{var}_t^T[x(T)] = V_T(t) \equiv -\beta(T) - \frac{\ddot{B}_T(t)}{2\dot{B}_T(t)} \quad (= -\beta(T) - \frac{1}{2} \frac{\partial}{\partial T} \log \dot{B}_T(t)) \quad (8.1)$$

$$E_t^T[x(T)] = \sqrt{\dot{B}_T(t)}x(t) + \dot{b}_T(t)/\sqrt{\dot{B}_T(t)} \quad (8.2)$$

$$\dot{c}_T(t) = \frac{1}{2} \sum_{i=1}^n [(\dot{b}_T^i(t))^2 / \dot{B}_T^i(t) + V_T^i(t)] \quad (8.3)$$

$$r(T)|_t^T \sim \Omega_n(1, 0, 0, \sqrt{\dot{B}_T(t)}x(t) + \dot{b}_T(t)/\sqrt{\dot{B}_T(t)}, V_T(t)) \quad (8.4)$$

$$-\log P_s(T)|_t^T \sim \Omega_n(B_s(T), b_s(T), c_s(T), \sqrt{\dot{B}_T(t)}x(t) + \dot{b}_T(t)/\sqrt{\dot{B}_T(t)}, V_T(t)) \quad (8.5)$$

$$B_s(T) = \frac{B_s(t) - B_T(t)}{\dot{B}_T(t) - V_T(t)(B_s(t) - B_T(t))} \quad (8.6)$$

$$\dot{B}_s(T) = \frac{\dot{B}_s(t)\dot{B}_T(t)}{(\dot{B}_T(t) - V_T(t)(B_s(t) - B_T(t)))^2} = \frac{\dot{B}_s(t)\dot{B}_T(t)B_s^2(T)}{(B_s(t) - B_T(t))^2} \quad (8.7)$$

$$b_s(T) = B_s(T)\sqrt{\dot{B}_T(t)}\left(\frac{b_s(t) - b_T(t)}{B_s(t) - B_T(t)} - \frac{\dot{b}_T(t)}{\dot{B}_T(t)}\right) \quad (8.8)$$

$$\dot{b}_s(T) = [\dot{b}_s(t)(1 + V_T(t)B_s(T)) + \dot{B}_s(T)(V_T(t)(b_s(t) - b_T(t)) - \dot{b}_T(t))]/\sqrt{\dot{B}_T(t)} \quad (8.9)$$

$$c_s(T) = c_s(t) - c_T(t) - \tilde{c}(B_s(T), b_s(T), \dot{b}_T(t)/\sqrt{\dot{B}_T(t)}, V_T(t)) \quad (8.10)$$

$$\partial V_T(t)/\partial t + \sigma^2(t)\dot{B}_T(t) = 0 \quad (8.11)$$

$$\alpha(u) = (\dot{B}_u(t))^{-3/2}(\dot{B}_u(t)\ddot{b}_u(t) - \ddot{B}_u(t)\dot{b}_u(t)) (= \ddot{b}_u(u)) \quad (8.12)$$

$$\sigma^2(u) = \dot{V}_u(t) + V_u^2(t) + 2\beta(u)V_u(t) (= \dot{V}_u(u)) \quad (8.13)$$

$$\beta'(u) + \beta^2(u) + \sigma^2(u) + \frac{1}{2} \frac{\partial^2}{\partial u^2} \log(\dot{B}_u(t)) - \frac{1}{4} \left(\frac{\partial}{\partial u} \log \dot{B}_u(t)\right)^2 = 0, \quad \beta(t) = -\frac{1}{2} \dot{B}_t(t) \quad (8.14)$$

$$E_t^s[x(T)] = \sqrt{\frac{\dot{B}_s(t)}{\dot{B}_s(T)}}x(t) + \frac{\dot{b}_s(t)}{\sqrt{\dot{B}_s(t)\dot{B}_s(T)}} - \frac{\dot{b}_s(T)}{\dot{B}_s(T)}, \quad \text{var}_t^s[x(T)] = \frac{V_s(t) - V_s(T)}{\dot{B}_s(T)} \quad (8.15)$$

 Note that the Jamshidian equation for h_{τ} has "... + gamma + beta)" which is incorrect, it should be "... + gamma - beta)"
 Compare with James / Webber, Equation 9.68 which is Jamshidian's $BT(t)$ containing h_{τ}

Moreover, if σ and β are constants, then

$$V_T(t) = \sigma^2 B_T(t), \quad B_T(t) = h(\tau)(e^{2\tau} - 1), \quad \dot{B}_T(t) = (2\gamma h(\tau)e^{\tau})^2$$

$$h(\tau) \equiv ((\gamma + \beta)e^{2\tau} + \gamma + \beta)^{-1}, \quad \gamma \equiv \sqrt{\sigma^2 + \beta^2}, \quad \tau \equiv T - t$$

If in addition α is constant, then (setting $\Sigma x \equiv x_1 + \dots + x_n$ for any vector x),

$$b_T(t) = \alpha h(\tau)(e^{\tau} - 1)^2/\gamma, \quad \dot{b}_T(t) = 2\alpha h^2(\tau)e^{\tau}(e^{\tau} - 1)/h(\tau/2)$$

$$c_T(t) = \sum \frac{1}{2} \left(\frac{\alpha^2}{\gamma^2} - \frac{\sigma^2}{\gamma - \beta} \right) \tau - \frac{1}{2} \log(2\gamma h(\tau))$$

$$+ \frac{\alpha^2}{\gamma^3(\gamma + \beta)} \left(h(\tau)(2\beta(\gamma + \beta)e^{\tau} + \sigma^2 - \beta^2) - \beta - \frac{\gamma}{2} \right)$$

$$\dot{c}_T(t) = \sum \frac{1}{2} \left(\frac{\alpha h(\tau)(e^{\tau} - 1)}{\gamma h(\tau/2)} \right)^2 + \frac{1}{2} \sigma^2 h(\tau)(e^{2\tau} - 1), \quad \tau \equiv T - t$$

Equations (8.1) and (8.2) provide formulae for the variance and mean of the state variable $X(T)$ in the T -maturity forward risk adjusted measure. Since $X(T)$ is normally distributed, its distribution is thus determined. Combined with the fact that $r(T)$ and logarithms of zero-coupon bond prices are quadratic in $X(T)$, their distribution is now determined as in equations (8.4) and (8.5). The option- and futures-pricing formulae of Section 4 are thus applicable. For options on forward bond contracts, we need the distribution of $X(T)$ in the s -maturity forward-risk-adjusted measure, $s \geq T$. This is provided by equation (8.15).

Equation (8.3) plays an important role in fitting the yield curve, as explained in the next section. Equations (8.6)–(8.10) are 'transport equations'. These equations provide the coefficient of discount factors and instantaneous rates at time T in terms of those at an earlier time t . In order to price an option in terms of the initial yield curve data, they must be substituted in equation (8.5). They are also useful for numerical evaluation of options on a finite difference grid.

Equations (8.12)–(8.14) are 'forward' ODEs for the coefficients of the state variable $x(t)$. As explained in the next section, these equations are useful for constructing the process for $x(t)$ from initial yield curve and/or volatility data for the purpose of American option valuation. These forward ODEs are related to the Fokker–Plank forward equation for Green's function, and in fact follow from the latter with the aid of (3.1), (8.1) and (8.2). Equation (8.11) is a useful 'backward' ODE for the variance function.

The final part of the theorem provides a closed-form solution for bond prices when the coefficients of $x(t)$ are time independent. An examination of the formulae actually reveals a disappointing feature of the quadratic model. Consider the single-factor case for simplicity, and assume that the drift parameter α is positive. The problem occurs when the state variable x is in the range $-\alpha/\gamma < x \leq 0$. Since $\alpha > 0$, this may not be very likely, but it is certainly possible. When $x(t)$ is in this range, the formulae show that while yields of short-date bonds have an instantaneous correlation of 1 with the spot interest rate $r(t)$, yields of long-dated bonds have instantaneous

correlation of -1 with the spot rate. In other words if the short rate moves up over a very short period of time, then long rates of necessity move down. This is clearly an undesirable property that is not shared by well-behaved models like the simple square-root model, or other specific models in which $r(t)$ is a diffusion (Markov) process.

To see this claim, note that bond price volatilities are given by

$$\sigma_T(t) = -\sigma(B_T(t)x(t) + b_T(t)) = -\sigma h(\tau)(e^{\tau} - 1)((e^{\tau} + 1)x(t) + \alpha(e^{\tau} - 1)/\gamma)$$

This quantity is zero when $x(t) = \alpha(1 - e^{\tau})/\gamma(1 + e^{\tau})$. Now, given $-\alpha/\gamma < x \leq 0$, define

$$\tau(x) \equiv \frac{1}{\gamma} \log \left(\frac{\alpha - \gamma x}{\alpha + \gamma x} \right) \quad \left(x = \frac{\alpha(1 - e^{\tau(x)})}{\gamma(1 + e^{\tau(x)})} \right)$$

Then, by what we have just seen the volatility of the zero-coupon bond of maturity $T^* = t + \tau(x(t))$ is zero, when the model at time t is at state $x(t)$. Substituting the expression for $x = x(t)$ from the previous equation into the one before, we easily see that for maturities $T < T^*$, the bond volatility is positive, and for maturities $T > T^*$ it is negative, when the model is in state $x(t)$. This establishes the claim.

We also note that for fixed t and T , the T -maturity zero-coupon bond price $P_T(t)$ reaches a maximum (over all states) at the state $x(t) = \alpha(1 - e^{\tau})/(\gamma(1 + e^{\tau}))$. At this state $dP_T(t)/P_T(t) = r(t) dt$, because the bond volatility is zero. But the large negative convexity should be a concern to the holder of this bond at (or near) this state. Such anomalies persist in state variable models of the form $r(t) = r(t, x(t))$ where the function $r(t, x)$ is not monotone in x . But if $r(t, x)$ is monotone in x , then $r(t)$ will be a diffusion process, and such a model reduces to the usual one-factor model where $r(t)$ serves as the state variable.

For the purpose of calculating futures prices, we need the distribution of the state variables and bond prices in the risk-neutral measure. The following theorem provides this, as well as the distribution of futures prices in the forward-risk-adjusted measure for valuation of options on bond futures or the delivery option.

Theorem 8.2: For any $t \leq s \leq t^*$, (using vector notation) set

$$\beta_s(t) \equiv \exp \left(- \int_t^s \beta(u) du \right), \quad a_s(t) \equiv \int_t^s \alpha(u) \beta_s(u) du, \quad v_s(t) \equiv \int_t^s \sigma^2(u) \beta_s^2(u) du$$

$$B_{s,t^*}(t) \equiv \frac{B_{t^*}(s) \beta_s^2(t)}{1 + B_{t^*}(s) v_s(t)}, \quad b_{s,t^*}(t) \equiv \frac{(B_{t^*}(s) a_s(t) + b_{t^*}(s)) \beta_s(t)}{1 + B_{t^*}(s) v_s(t)}$$

$$c_{s,t^*}(t) \equiv c_{t^*}(s) + \tilde{c}(B_{t^*}(s), b_{t^*}(s), a_s(t), v_s(t))$$

Then, for all $0 \leq t \leq T \leq s \leq t^*$, the following equations hold:

$$E_t[x(s)] = \beta_s(t)x(t) + a_s(t), \quad \text{var}_t[x(s)] = v_s(t) \quad (8.16)$$

$$-\log P_{t^*}(s) |_t \sim \Omega_n(B_{t^*}(s), b_{t^*}(s), c_{t^*}(s), \beta_s(t)x(t) + a_s(t), v_s(t)) \quad (8.17)$$

$$F_{s,t^*}(t) \equiv E_t[P_{t^*}(s)] = \exp \left[- \sum_{i=1}^n \left(\frac{1}{2} B_{s,t^*}^i(t) x_i^2(t) + b_{s,t^*}^i(t) x_i(t) \right) - c_{s,t^*}(t) \right]$$

$$-\log F_{s,t^*}(T) |_t^T \sim \Omega_n \left(B_{s,t^*}(T), b_{s,t^*}(T), c_{s,t^*}(T), \sqrt{\dot{B}_T(t)} x(t) + \dot{b}_T(t) / \sqrt{\dot{B}_T(t)}, V_T(t) \right)$$

$$B_{s,t^*}(T) = \frac{B_{s,t^*}(t)}{\beta_T^2(t) - v_T(t) B_{s,t^*}(t)}, \quad b_{s,t^*}(T) = B_{s,t^*}(t) \left(\frac{b_{s,t^*}(t)}{B_{s,t^*}(t)} \beta_T(t) - a_T(t) \right)$$

$$c_{s,t^*}(T) = c_{s,t^*}(t) - \tilde{c}(B_{s,t^*}(T), b_{s,t^*}(T), a_T(t), v_T(t))$$

Moreover, if α , β , and σ are constants, then

$$\beta_s(t) = e^{-\beta(s-t)}, \quad a_s(t) = \alpha(1 - e^{-\beta(s-t)})/\beta, \quad v_s(t) = \sigma^2(1 - e^{-2\beta(s-t)})/2\beta$$

9. Fitting initial yield and volatility curves and preference-free evaluation

Consider first the one-factor case. Assume that at the initial time t (e.g. $t = 0$) the yield curve is given, that is $P_T(t)$ (and $r_T(t)$) is given for all $T \geq t$. We know $x(t) = \pm(2r(t))^{1/2}$, but we do not know its sign. Assume its sign is also given, say positive. We first consider the problem of fitting only the initial curve. For this it is natural to assume that both $\beta(u)$ and $\sigma(u)$ are given for all $u \geq t$, and solve for $\alpha(u)$ for all u , or solve directly for the coefficients of the discount function.

So suppose $\beta(u)$ and $\sigma(u)$ are given for all $u \geq t$. A natural case is when they are constants. In that case we have closed-form solutions for $B_s(T)$, $\dot{B}_s(T)$ and $V_T(t)$ from the second part of Theorem 8.1. If calibration to caps or swaptions is desired, $\beta(u)$ or $\sigma(u)$ may be chosen time dependent. In that case, we numerically solve (in linear time) the single ODE (8.14) (or solve ODE (8.13) and then use (8.1)). Then $\dot{B}_T(t)$, $B_T(t)$ and $V_T(t)$ are determined (at least for T near t). Once these quantities are found, either in closed form or by solving an ODE as the case may be, then using equation (8.3) for $\dot{c}_T(t)$, we write

$$r_T(t) = \frac{1}{2} \dot{B}_T(t) x^2(t) + \dot{b}_T(t) x(t) + \frac{1}{2} [(\dot{b}_T(t))^2 / \dot{B}_T(t) + V_T(t)]$$

With the initial curve $r_T(t)$ at time t given, this is a quadratic equation for $\dot{b}_T(t)$. Its (largest) solution is given by

$$\dot{b}_T(t) = -\dot{B}_T(t) x(t) + \sqrt{\dot{B}_T(t)(2r_T(t) - V_T(t))} \quad (2r_T(t) > V_T(t)) \quad (9.1)$$

This, together with equation (8.12), furnishes an algebraic formula for the unknown $\alpha(u)$. This can

then be used to build a finite difference grid for options that require numerical valuation. We note that for the solution in (9.1) to exist, we must have $V_T(t) \leq 2r_T(t)$. Otherwise, a real-valued solution does not exist, that is, the yield curve does not fit. Also note that a positive solution obtains if and only if $2r_T(t) > \dot{B}_T(t)x^2(t) + V_T(t)$.

Once $\dot{b}_T(t)$ is determined from (9.1), the distribution of $r(T)$ is available from (8.4), so spot rate caps are priced by Theorem 4.1 (ii). Further, from $\dot{b}_u(t)$, we recover $\dot{c}_u(t)$ from (8.3) and with it $c_u(t)$. So, we have all the bond coefficients ($B_s(t)$, etc.), for the initial time t are now available. Moreover, equations (8.6)–(8.10) now algebraically provide them ($B_s(T)$, etc.) for time T . In particular we now have the distribution of $P_s(T)$ from (8.5), with which to analytically evaluate the various bond options in Theorem 4.1.

We have thus seen that if $\beta(u)$ and $\sigma(u)$ are constant, the fitting of the curve is entirely analytic via closed form algebraic formula, and when they are not constant, only a single ODE need be numerically solved. The resulting valuation of contingent claims is preference free, that is, it is in terms of the initial yield curve and does not require or use any specification of the market price of risk.

Suppose next that both the initial forward-rate curve $r_T(t)$ and initial forward-rate volatility curve $\dot{\sigma}_T(t) \equiv \partial \sigma_T(t) / \partial T$ are given at the initial time t . Now, $\dot{\sigma}_T(t) = (\dot{B}_T(t)x(t) + \dot{b}_T(t))\sigma(t)$. Comparison with (9.1) gives

$$\delta_T(t) \equiv \left(\frac{\dot{\sigma}_T(t)}{\sigma(t)} \right)^2 = \dot{B}_T(t)(2r_T(t) - V_T(t)) = \dot{B}_T(t)(\beta(T) + 2r_T(t)) + \frac{\dot{B}_T(t)}{2} \quad (9.2)$$

(Note, $\delta_T(t)$ is known because $\sigma(t) = \dot{\sigma}_t(t)/x(t)$ is known.) Solving for $\dot{B}_T(t)$, we obtain

$$\dot{B}_T(t) = ((P_T(t))^2 \beta_T(t))^2 \left(1 + 2 \int_t^T \frac{\delta_u(t) du}{((P_u(t))^2 \beta_u(t))^2} \right) \left(\beta_s(t) \equiv \exp \left(- \int_t^s \beta(u) du \right) \right)$$

Moreover, substituting for $\beta(T)$ from (9.2) into (8.14) results in

$$2r\dot{B}\ddot{B} - 2\delta\dot{B} + (-2\dot{r} + 4r^2 + \sigma^2(T))\dot{B}^2 + (\dot{\delta} - 4r\delta)\dot{B} + \delta^2 = 0, \quad r \equiv r_T(t), \quad \dot{B} \equiv \dot{B}_T(t), \quad \text{etc}$$

Now, there are two possibilities: either $\beta(u)$ is given or $\sigma(u)$ is given. Assume first that $\beta(u)$ is given (e.g. a constant). Then $\dot{B}_T(t)$ can be numerically computed linearly in time from the equation before the last equation above (first compute and store $\beta_u(t)$ for all $t \leq u \leq T$), and $b_T(t)$ is subsequently obtained from (9.1). Moreover, $\sigma^2(u)$ is recovered from (8.13) (and will be positive near t by continuity.) Option evaluation now proceeds as before, and it will be preference free if the market price of risk is deterministic.

More interestingly, if $\sigma(u)$ is given, then the above equation is a first order ODE for $\dot{B}_T(t)$ (as a function of T), which, subject to $\dot{B}_t(t) = 1$, can be numerically solved for all T (near t) in linear time. Then $\beta(u)$ is recovered from (9.2). Now, evaluation is preference free whatever the market price of risk, for the latter does not affect $\sigma(u)$ in the diffusion term.

Thus, to fit both the yield and volatility curves, a single ODE needs be solved numerically. It is interesting to contrast this with simple square-root models in Jamshidian (1995). There, yield and volatility curves are simultaneously fitted analytically, but to fit only the yield curve, an ODE must be solved numerically. In this way, quadratic and simple square-root models exhibit opposite behaviour.

As preparation for the multifactor case, given the model parameters $\alpha(u)$, $\beta(u)$, and $\sigma(u)$, consider the problem of determining $r_T(t)$, $P_T(t)$, $B_T(t)$, etc., at a fixed initial time t for 'all' (i.e. many) T . If they are constants, of course we use the closed-form solutions in the second part of Theorem 7.1. If not, we could use the backward ODEs which follow from the fundamental PDE. But then, for each T , we must solve one ODE. Instead, we should numerically solve once and for all the forward ODE (8.14) (or (8.13)) for $\hat{B}_u(t)$ for all u , and then solve the forward ODE (8.12) for $\hat{b}_u(t)$ for all u , both linearly in time. From (8.3), $\hat{c}_u(t)$ is now also determined so that we have $r_T(t)$ in linear time. As a bonus, we integrate these to find in linear time $B_u(t)$, $b_u(t)$ and $c_u(t)$, and then use the forward algebraic equations (8.6)–(8.10) to find $B_s(T)$, etc. conveniently for any s and T .

Now consider the multifactor case $n \geq 2$. Say, we are given $r_T(t)$ at the initial time t and the vectors $\beta(u)$ and $\sigma(u)$ (perhaps constants). There are now infinitely many compatible quadratic models because we have n degrees of freedom $\alpha_1(u), \dots, \alpha_n(u)$ but only one set of data to fit. One way to fix a unique choice is to take, say, $\alpha_2(u), \dots, \alpha_n(u)$ as given. In this case, we consider the one-factor models corresponding to $i = 2, \dots, n$, whose α , β , and σ coefficients are given, and calculate the corresponding forward-rate curve for each, denoted $r_T^i(t)$, as described in the previous paragraph. Then, we set $r_T^1(t) = r_T(t) - r_T^2(t) - \dots - r_T^n(t)$, and find, as explained above, a one-factor model, together with its $\alpha_1(t)$, $\beta_1(t)$, etc., that fits $r_T^1(t)$, $\beta_1(u)$, and $\sigma_1(u)$. The resulting multifactor model now fits the original $r_T(t)$, and we have its appropriate coefficients. To additionally fit a volatility curve, we solve for $\hat{B}_T(t)$ in terms of $r_T^1(t)$, $\sigma_1(t)$, and either $\beta_1(u)$ or $\sigma_1(u)$ as before in the one-factor case. Another way to deal with the redundancy is to break $r_T(t)$ arbitrarily into n components $r_T^1(t), \dots, r_T^n(t)$, (e.g. $r_T^i(t) = r_T(t)/n$) and fit a one-factor model through each.

When the number of factors is greater than two, it may be preferable to choose a constant coefficient model. In this case there are sufficiently many coefficients available to calibrate the model to a handful of key rates. Using the closed-form solutions for bond prices as given in Theorem 8.1, the multidimensional numerical root search should be fast.

10. Conclusion

The quadratic model is in many ways similar to the class of simple square-root models of Jamshidian (1995). In both models interest rates are chi-squared distributed, bond prices can be solved by the Riccati equations, closed-form solutions are available for futures and options on zero-coupon bonds, and above all, the yield curve can be fitted analytically.

In our opinion, simple square-root models are better, however. One reason is that, at least for the single-factor case, the spot rate $r(t)$ serves as the state variable in simple square-root models, and zero-coupon bond prices are decreasing prices of this state variable. One can consequently evaluate a European option on a coupon bond analytically as a linear combination of European options on zero-coupon bonds. But for the quadratic model, bond prices are a non-monotone function of the state variable, and this decomposition does not seem possible.

Another reason is the anomalous behaviour of the quadratic model when the state variable is negative. In Section 7 we saw that the spot rate can reach and bounce off zero, and given enough time this could happen several times. More seriously, we saw in Section 8 that for a certain range of parameters, long-term rates will of necessity be negatively correlated with short-term rates. These and other related problems are not as severe as the negative interest-rates associated with Gaussian

interest-rate models. But, compared to simple square-root models, which do not suffer from any such anomalies, the quadratic interest rate-model seems to have less to offer.

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Appendix A

Theorem 2.1 is a consequence of the following general result which leads to similar explicit formulae for any distribution whose characteristic function has an elementary form.

Theorem 11.1: Let X be a (real) random variable and denote its probability distribution and

density functions respectively by $P(x)$ and $p(x) = P'(x)$ and its characteristic function by $\phi(\xi) = E[\exp(i\xi X)]$, $i = \sqrt{-1}$, $\xi \in \mathbb{R}$. Assume $\phi(\xi)$ is smooth and $|\phi(\xi)| < C/(1+|\xi|)^\epsilon$ for some positive constants C and ϵ . Then, for all $x \in \mathbb{R}$ ¹¹

$$P(x) = \frac{1}{2} + \frac{1}{2\pi i} \int_0^\infty (e^{i\xi x} \phi(-\xi) - e^{-i\xi x} \phi(\xi)) \frac{d\xi}{\xi} \quad (i = \sqrt{-1}) \quad (A1)$$

$$E[(X-x)_+] = \frac{1}{2}(E[X] - x) + \frac{1}{2\pi} \int_0^\infty (2 - e^{i\xi x} \phi(-\xi) - e^{-i\xi x} \phi(\xi)) \frac{d\xi}{\xi^2} \quad (A2)$$

Proof: Let $H(x)$ denote the Heaviside function, i.e. $H(x) = 1$ if $x > 0$, $H(x) = 0$ if $x \leq 0$. For any ordinary or generalized function g , denote its Fourier transform by \hat{g} . Then, by the Fourier inversion formula, $p(x) = \hat{\phi}(x)/2\pi$. Hence, denoting the duality between a generalized function u and a test function φ by $\langle u, \varphi \rangle$, we have,

$$P(x) = \int_{-\infty}^x p(y) dy = \frac{1}{2\pi} \int_{-\infty}^\infty H(x-y) \hat{\phi}(y) dy = \frac{1}{2\pi} \langle \hat{H}(\xi), e^{ix\xi} \phi(-\xi) \rangle$$

Now, it is well known that as a generalized function \hat{H} is given by¹²

$$\hat{H}(\xi) = \pi \delta(\xi) - i \frac{d}{d\xi} \log |\xi|$$

where δ is the Dirac delta function. Thus, for any test function φ ,¹³

$$\langle \hat{H}(\xi), \varphi(\xi) \rangle = \pi \varphi(0) + i \int_{-\infty}^\infty \log |\xi| \varphi'(\xi) d\xi = \pi \varphi(0) + i \int_0^\infty \frac{\varphi(-\xi) - \varphi(\xi)}{\xi} d\xi$$

Equation (A1) follows. Next, since $y_+ = yH(y)$, we have, $\hat{y}_+(\xi) = i d\hat{H}/d\xi$. Hence,

$$E[(X-x)_+] = \frac{1}{2\pi} \int_{-\infty}^\infty (y-x)_+ \hat{\phi}(y) dy = \frac{i}{2\pi} \left\langle \frac{d}{d\xi} \hat{H}(\xi), e^{-ix\xi} \phi(\xi) \right\rangle$$

Now, for any test function φ ,

$$\begin{aligned} \left\langle \frac{d}{d\xi} \hat{H}(\xi), \varphi(\xi) \right\rangle &= -\langle \hat{H}(\xi), \varphi'(\xi) \rangle = -\pi \varphi'(0) - i \int_0^\infty \frac{\varphi'(-\xi) - \varphi'(\xi)}{\xi} d\xi \\ &= -\pi \varphi'(0) - i \int_0^\infty \frac{2\varphi(0) - \varphi(\xi) - \varphi(-\xi)}{\xi^2} d\xi \end{aligned}$$

Equation (A2) follows because if $\varphi(\xi) = \exp(-i\xi x) \phi(\xi)$, then $\varphi'(0) = i(E[X] - x)$. \square

¹¹ Note that both integrands are integrable (in fact smooth) at $\xi = 0$, and they are integrable at infinity by the assumption on the growth of $\phi(\xi)$.

¹² See, e.g. Hörmander (1983), Example 7.1.17. Briefly, a generalized function $u(x)$ is a continuous linear functional on appropriate spaces of 'test functions' $\varphi(x)$. The largest domain that $u(x)$ continuously extends to depends on its degree (of non-differentiability) and its growth. By definition, $\langle u, \varphi \rangle \equiv -\langle u, \varphi' \rangle$, $\langle \hat{u}, \varphi \rangle \equiv \langle u, \hat{\varphi} \rangle$, and for a function $f(x)$, $\langle fu, \varphi \rangle \equiv \langle u, f\varphi \rangle$. If $u(x)$ is an integrable function, then $\langle u, \varphi \rangle = \int_{-\infty}^\infty u(x) \varphi(x) dx$.

¹³ The second integral here is known as a principal value integral. It is obtained by writing the first integral as $\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^\infty$, integrating by parts, and letting ϵ approach zero. The similar integral below is another principal value integral and is obtained similarly.

Remark 11.1: Given a scalar β , under a slightly stronger assumption, one shows similarly,

$$\int_{-\infty}^x p(y) e^{-\beta y} dy = \frac{1}{2} E[e^{-\beta X}] + \frac{1}{2\pi i} \int_0^\infty (e^{i\xi x} \phi(-\xi + i\beta) - e^{-i\xi x} \phi(\xi + i\beta)) \frac{d\xi}{\xi}$$

Differentiating with respect to β , at $\beta = 0$, and combining with (A1) gives again (A2).

Proof of Theorem 2.1: Apply Theorem 11.1 to the random variable $Y - \alpha$. Then, $\phi(\xi) = \Psi_n(\nu, \lambda, -i\xi\beta, 0)$, where the principal branch of the logarithm is taken, i.e.

$$-\frac{\pi}{2} < \text{Im}(\log(1 + iy)) = \arctan(y) < \frac{\pi}{2}$$

Now, calculating respectively the imaginary part of the integrand in (A1) and the real part of the integrand in (A2) gives respectively the first and the second equations in Theorem 2.1. The third equation in Theorem 2.1 is a special case of the fourth with $\beta' = 0$, and the fifth equation follows from the fourth since $\min(a, b) = a - (a - b)_+$ and $1 - \chi_n^2(y, \nu, \lambda, \beta) = \chi_n^2(-y, \nu, \lambda, -\beta)$. As for the fourth equation, we have

$$E[(e^{-Y} - e^{-Y'})_+] = \int_{\mathcal{R}} (f(x; \beta, \alpha) - f(x; \beta', \alpha')) dx \quad (A.3)$$

$$f(x; \beta, \alpha) \equiv e^{-\alpha} \prod_{j=1}^n e^{-\beta_j x_j} p(x_j; \nu_j, \lambda_j), \quad \mathcal{R} \equiv \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n (\beta_j - \beta'_j) x_j \leq \alpha - \alpha' \right\}$$

where $p(x; \nu, \lambda)$ denotes the non-central χ^2 density function, a property of which yields

$$f(x; \beta, \alpha) = E[e^{-Y}] \prod_{j=1}^n (1 + 2\beta_j) p\left((1 + 2\beta_j)x_j; \nu_j, \frac{\lambda_j}{1 + 2\beta_j}\right)$$

Inserting in (A3), changing variables to $x' = (1 + 2\beta)x$, and doing the same for $f(x; \beta', \alpha')$ now yields the desired result.

Remark 11.2: The third, fourth and fifth equations in Theorem 2.1 can also be derived from the general equation in Remark 9.1 and a change of variable.

Proof of Theorem 8.1: Taking $\partial/\partial T$ of both sides of the backward ODE for $B_T(t)$, we obtain

$$\frac{d}{dt} \log(\dot{B}_T(t)) = 2(\beta(t) + \sigma^2(t) B_T(t)) \quad (A4)$$

Taking $\partial/\partial T$ of both sides gives (8.11). Equation (A4) also implies

$$0 = \frac{d}{dt} 1 = \frac{d}{dt} \ddot{B}_t(t) = \dot{B}_t(t) + \frac{d\dot{B}_T(t)}{dt} \Big|_{T=t} = \ddot{B}_t(t) + 2\beta(t)$$

Hence, $V_t(t) = 0$. Equation (A4) (using $\dot{B}_T(T) = 1$) also gives

$$\exp\left(-2 \int_s^T (\beta(u) + \sigma^2(u) B_T(u)) du\right) = \dot{B}_T(s)$$

Using this and the process for $x(t)$ in the T -maturity forward-risk-adjusted measure, we obtain

$$\text{var}_t^T[x(T)] = \int_t^T \sigma^2(s) \dot{B}_T(s) ds$$

By (8.11) and the fact that $V_t(t) = 0$, equation (8.1) now follows. Next,

$$\sum_{j=1}^n (\frac{1}{2} \dot{B}_T^j(t) x_j^2(t) + \dot{b}_T^j(t) x_j(t)) + \dot{c}_T(t) = r_T(t) = E_t^T[r(T)] = \frac{1}{2} \sum_{j=1}^n ((E_t^T[x_j(T)])^2 + V_T^j(t))$$

On the other hand we know $E_t^T[x_j(T)]$ is linear in $x_j(t)$. Setting powers of $x_j(t)$ in the above equation equal, we obtain (8.2) and (8.3). As a bonus we also obtain

$$\frac{\dot{b}_T(t)}{\sqrt{\dot{B}_T(t)}} = \int_t^T (\alpha(u) - \dot{b}_T(u)) \sqrt{\dot{B}_T(u)} du$$

This also follows directly from the ODE for $b_T(t)$ and (A4). Indeed, more generally,

$$\frac{\dot{b}_s(t)}{\sqrt{\dot{B}_s(t)\dot{B}_s(T)}} - \frac{\dot{b}_s(T)}{\dot{B}_s(T)} = \int_t^T (\alpha(u) - \dot{b}_s(u)) \sqrt{\frac{\dot{B}_s(u)}{\dot{B}_s(T)}} du$$

from which the more general equation (8.15) follows. Equations (8.4) and (8.5) are immediate consequences of (8.1) and (8.2). To derive (8.6)–(8.10), we use forward risk adjustment, namely equation (3.2). Indeed, the right-hand side of (3.2) can be written as a quadratic function of $x(t)$ using (6.1). Setting powers of x of both sides of (3.2) equal and rearranging gives (8.6), (8.8) and (8.10). Equations (8.7) and (8.9) follow by differentiating respectively (8.6) and (8.8) and rearranging. Equation (8.14) obtains by substituting the definition of $V_T(t)$ in (8.13). We give two derivations for (8.12) and (8.13). Using (8.11) twice,

$$\frac{\partial}{\partial t} (\dot{V}_u(t) + V_u^2(t) + 2\beta(u)V_u(t)) = -\sigma^2(t)\ddot{B}_T(t) - 2(\sigma^2(t)\dot{B}_T(t))(V_T(t) + \beta_T(t)) = 0$$

where the last equality used the definition of $V_T(t)$. In addition, using (8.11) again,

$$0 = \frac{d}{dt} V_t(t) = \dot{V}_t(t) + \frac{\partial V_T(t)}{\partial t} \Big|_{T=t} = \dot{V}_t(t) - \sigma^2(t)$$

Hence equation (8.13) is valid at $t = u$, and since we just showed that its right-hand side is independent of t , equation (8.13) is valid everywhere. Equation (8.12) follows by a similar argument. Alternatively, we use the following argument. In general, let $\mu(t)$ denote the risk-neutral drift of $r(t)$. Taking total derivative of the equation $r(t) = r_t(t)$, we can easily show that $\mu(t) = \partial r_T(t) / \partial T \Big|_{T=t}$. In the quadratic model, using Ito's lemma, this yields,

$$\begin{aligned} \sum_{j=1}^n (\frac{1}{2} \ddot{B}_t^j(t) x_j^2(t) + \ddot{b}_t^j(t) x_j(t)) + \ddot{c}_t(t) &= \dot{r}_t(t) = \mu(t) \\ &= \sum_{j=1}^n [x_j(t)(\alpha_j(t) - \beta_j(t)x_j(t)) + \frac{1}{2}\sigma_j^2(t)] \end{aligned}$$

Setting powers of x equal, we obtain:

$$\beta(t) = -\frac{1}{2}\ddot{B}_t(t), \quad \alpha(t) = \ddot{b}_t(t), \quad \sum_{j=1}^n \sigma_j^2(t) = 2\ddot{c}_t(t)$$

(These also follow from the backward ODEs.) Now, differentiating (8.9) we obtain:

$$\alpha(u) = \ddot{b}_u(u) = (\dot{B}_u(t))^{-1} [\ddot{b}_u(t) + 2\dot{b}_u(t)(V_u(t) + \beta(u))]$$

which simplifies to (8.12). Differentiating (8.7) gives similarly (the second derivation of) (8.13). (Other derivations of (8.12) and (8.13) include using the Fokker–Plank forward equation or the equation $\partial r_T(t) / \partial T = E_t^T[\mu(T)] - \text{var}_t^T[r(T)]$.) Finally, the correctness of the formulae for the constant coefficient case can be verified directly by substituting into the backward ODEs, although the above general results do provide shortcuts.

Proof of Theorem 8.2: Equation (8.16) is a known property of Gaussian processes. Equation (8.17) is now immediate. The futures-pricing formula now follows from (8.17) and (6.1). The next equation is now immediate from (8.16), (8.1) and (8.2). The following three equations follow by writing the right-hand side of the equation $F_{s,r}(t) = E_t[F_{s,r}(T)]$, with the aid of (8.16) and (6.1), as a quadratic function of $x(t)$, then setting powers of x equal to those of the left-hand side, and rearranging the results. The equations for the constant coefficient case are clear.

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Student's name	Gerhard Rebel
Degree programme	ISMA MSc ISIB
Year	Masters
Type of work	Numerical Methods 2 Project
Title of module	Numerical Methods 2
Name of lecturer/tutor	Ubbo Wiersema
Submission deadline	1 June 2004
This piece of work was undertaken	Independently

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